Conditional Expectations and Variances

Andrew Nobel

October, 2024

Conditional Expectation

Conditional Expectation

Let (X, Y) be jointly distributed with $X \in \mathcal{X}$ and $Y \in \mathbb{R}$. Assuming $\mathbb{E}|Y|$ is finite, let

$$\varphi(x) = \begin{cases} \sum_y y\, p(y|x) & \text{if } Y \text{ is discrete} \\ \int y\, f(y|x)\, dy & \text{if } Y \text{ is continuous} \end{cases}$$

Define the following

- $\mathbb{E}(Y|X = x) = \varphi(x)$, the conditional expectation of Y given $X = x$ (a number)
- $\mathbb{E}(Y|X) = \varphi(X)$, the conditional expectation of Y given X (a random variable)

Note: Definition readily extends to more complex settings, e.g.,

$$\mathbb{E}(g(Y, Z)|X = x) = \sum_{y,z} g(y, z) p(y, z|x)$$

Properties of Conditional Expectation

Provided all the relevant expectations are well-defined

1. Order. If $Y \leq Z$ then $\mathbb{E}(Y|X) \leq \mathbb{E}(Z|X)$ with probability one
2. Law of total expectation. $\mathbb{E}\{\mathbb{E}(Y|X)\} = \mathbb{E}Y$
3. Linearity. $\mathbb{E}(aZ + bY|X) = a\mathbb{E}(Z|X) + b\mathbb{E}(Y|X)$
4. Independence. If X, Y are independent then $\mathbb{E}(Y|X) = \mathbb{E}(Y)$
5. Functions of X act like constants. $\mathbb{E}[f(X)g(Y)|X] = f(X)\,\mathbb{E}(g(Y)|X)$
6. Jensen's Inequality. If $g : \mathbb{R} \to \mathbb{R}$ is convex then $g(\mathbb{E}(Y|X)) \leq \mathbb{E}(g(Y)|X)$

Conditional Expectation and Prediction

Setting: Let (X, Y) be jointly distributed with $X \in \mathcal{X}$ and $Y \in \mathbb{R}$. Suppose that we wish to predict Y by a function of X

Fact: For any function $h : \mathcal{X} \to \mathbb{R}$, we have

$$\mathbb{E}(Y - h(X))^2 \; \geq \; \mathbb{E}(Y - \mathbb{E}(Y|X))^2$$

Upshot: Among all functions of X the conditional expectation $\varphi(x) = \mathbb{E}(Y|X = x)$ minimizes the mean squared prediction error of Y

Conditional Variance

Conditional Variance

Definition: Let (X, Y) be jointly distributed with $Y \in \mathbb{R}$. The conditional variance of Y given X is given by

$$\mathrm{Var}(Y|X) := \mathbb{E}\left[(Y - \mathbb{E}(Y|X))^2 | X\right]$$

Fact: The Conditional Variance Formula

$$\mathrm{Var}(Y) = \mathbb{E}\left[\mathrm{Var}(Y|X)\right] + \mathrm{Var}\left(\mathbb{E}(Y|X)\right)$$