STOR 893 Problems

1. Infima and Suprema

1. Let $A \subseteq \mathbb{R}$ be non-empty.

(a) $\inf(A) \leq x \leq \sup(A)$ for all $x \in A$

(b) $-\inf(A) = \sup(-A)$ and $-\sup(A) = \inf(-A)$

(c) If $A \subseteq B$ then $\inf(A) \geq \inf(B)$ and $\sup(A) \leq \sup(B)$

(d) For $b, c \in \mathbb{R}$ define $cA + b = \{ca + b : a \in A\}$. Show that $\sup(cA + b) = c \sup(A) + b$ and that $\inf(cA + b) = c \inf(A) + b$.

2. Let $\mathcal{X}$ be a non-empty set and let $f, g : \mathcal{X} \to \mathbb{R}$. Write $\sup f(x)$ for $\sup_{x \in \mathcal{X}} f(x)$ and $\inf f(x)$ for $\inf_{x \in \mathcal{X}} f(x)$.

(a) If $f \leq g$ then $\inf f(x) \leq \inf g(x)$ and $\sup f(x) \leq \sup g(x)$

(b) $\sup(f(x) + g(x)) \leq \sup f(x) + \sup g(x)$

(c) $\inf(f(x) + g(x)) \geq \inf f(x) + \inf g(x)$

(d) $|\sup f(x) - \sup g(x)| \leq \max |f(x) - g(x)|$

3. Let $A, B \subseteq \mathbb{R}$ be a non-empty set. Define $A + B = \{x + y : x \in A, y \in B\}$. Find equalities or inequalities relating the following quantities.

(a) $\sup(A + B)$ and $\sup(A) + \sup(B)$

(b) $\inf(A + B)$ and $\inf(A) + \inf(B)$

4. (Exchanging iterated maxima and iterated minima) Let $\mathcal{X}$ and $\mathcal{Y}$ be sets and let $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be any function.

(a) Show that

$$\sup_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} f(x, y) = \sup_{y \in \mathcal{Y}} \sup_{x \in \mathcal{X}} f(x, y)$$

(b) Show that

$$\inf_{x \in \mathcal{X}} \inf_{y \in \mathcal{Y}} f(x, y) = \inf_{y \in \mathcal{Y}} \inf_{x \in \mathcal{X}} f(x, y)$$

5. (Saddle points and minimax) Let $\mathcal{X}$ and $\mathcal{Y}$ be sets and let $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be any function.

(a) Show that, with no further assumptions,

$$\sup_{y \in \mathcal{Y}} \inf_{x \in \mathcal{X}} f(x, y) \;\leq\; \inf_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} f(x, y) \tag{1}$$

This simple fact plays an important role in optimization, where it implies the weak duality property of the Lagrange dual problem, and in game theory, where it has connections with Nash equilibria. A pair $(\tilde{x}, \tilde{y}) \in \mathcal{X} \times \mathcal{Y}$ is called a *saddle point* for f if

$$f(\tilde{x}, y) \;\leq\; f(\tilde{x}, \tilde{y}) \;\leq\; f(x, \tilde{y}) \quad \text{for every } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}$$

(b) Show that if $(\tilde{x}, \tilde{y})$ is a saddle point for f then

$$f(\tilde{x}, \tilde{y}) \;=\; \inf_{x \in \mathcal{X}} f(x, \tilde{y}) \quad \text{and} \quad f(\tilde{x}, \tilde{y}) \;=\; \sup_{y \in \mathcal{Y}} f(\tilde{x}, y)$$

To see how these inequalities explain the use of the terminology "saddle point", assume that f is nice and smooth, and sketch what it will look like in a neighborhood around the point $(\tilde{x}, \tilde{y})$.

(c) Show that the existence of a saddle point implies equality in inequality (1) above.

(d) Evaluate both sides of (1) when $\mathcal{X} = [0, 1]$, $\mathcal{Y} = [-1, 1]$, and $f(x, y) = x^2 y$.

6. Recall that if $f : \mathcal{X} \to \mathbb{R}$ is a real-valued function then the argmax of f is the set of points in x at which f is maximized,

$$\arg\max_{x \in \mathcal{X}} f(x) = \left\{ x \in \mathcal{X} : f(x) = \sup_{u \in \mathcal{X}} f(u) \right\}.$$

The argmin of f is similarly defined. Let $f : \mathcal{X} \to \mathbb{R}$ be defined on a set $\mathcal{X} \subseteq \mathbb{R}$ by $f(x) = x^2$. Identify the value of

$$\sup_{x \in \mathcal{X}} f(x) \quad \text{and} \quad \arg\max_{x \in \mathcal{X}} f(x)$$

in each of the following cases: $\mathcal{X} = [-2, 2]$, $\mathcal{X} = (-2, 2]$, $\mathcal{X} = (-2, 2)$, and $\mathcal{X} = (-3, 2]$.

7. Let A be a bounded subset of $\mathbb{R}^d$. Identify the values of $\inf f(x), \sup f(x), \arg\min f(x)$, and $\arg\max f(x)$ for the function $f : \mathbb{R}^d \to \mathbb{R}$ defined by

$$f(x) = \inf_{y \in A} ||x - y||.$$

8. Let $x_1, x_2, \ldots \in \mathbb{R}$ be a numerical sequence.

(a) Define what it means for x_n to converge to a number x

(b) Give a precise statement of what it means for x_n *not* to converge to x (i.e., the logical negation of the definition)

(c) Define what it means for x_n to converge to $+\infty$

(d) Give a precise statement of what it means for x_n not to converge to $+\infty$

(e) Give an example of a sequence that is unbounded above, but does not converge to $+\infty$

9. Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ be numerical sequences such that $x_n \to x$ and $y_n \to y$. Establish the following

(a) If a, b are constants then $ax_n + b \to ax + b$.

(b) $x_n + y_n \to x + y$

(c) $x_n y_n \to xy$

(d) $1/x_n \to 1/x$ if $x \neq 0$

10. Let $A \subseteq \mathbb{R}$ be non-empty. Show that there is a sequence $x_1, x_2, \ldots \in A$ such that x_n converges to $\sup(A)$. Consider separately the following cases.

(a) $\sup(A) \in A$, i.e., A is bounded above and the sup is achieved

(b) $\sup(A) < \infty$ but $\sup(A) \notin A$, i.e., A is bounded above and the sup is not achieved

(c) $\sup(A) = +\infty$, i.e., A is not bounded above

11. Let $x_1 \leq x_2 \leq \cdots \in \mathbb{R}$ be a non-decreasing sequence. Show that x_k converges to $\sup x_n$. Consider separately the case where $\sup x_n < \infty$ and the case where $\sup x_n = +\infty$.

12. Let $x_1, x_2, \ldots$ be a numerical sequence. Carefully define $\limsup_{n\to\infty} x_n$, which we will denote by $\overline{\lim}\, x_n$.

(a) Show that $\overline{\lim}\, x_n$ is the limit of a monotone sequence, and is therefore well defined, but possibly infinite.

(b) Give examples where $\overline{\lim}\, x_n = +\infty$ and $\overline{\lim}\, x_n = -\infty$.

(c) Show that $\overline{\lim}\, x_n = \inf_{n\geq 1} \sup_{m\geq n} x_m$.

(d) Define and establish similar results for $\liminf_{n\to\infty} x_n$, which we denote by $\underline{\lim}\, x_n$

2. Open, Closed, Compact Sets

1. Let (X, d) be a metric space.

   (a) If $K \subseteq X$ is compact then it is closed.

   (b) If $K \subseteq X$ is compact and $V \subseteq K$ is closed, then V is compact.

   (c) If V is closed and K is compact then $F \cap K$ is compact.

2. Let $\{K_\alpha : \alpha \in \Lambda\}$ be a family of compact subsets of a metric space (X, d). Suppose that for every finite set $\alpha_1, \dots, \alpha_k \in \Lambda$ the intersection $\cap_{j=1}^{k} K_{\alpha_j}$ is non-empty. Show that full intersection $\cap_{\alpha \in \Lambda} K_\alpha$ is non-empty.

3. If $K_1 \supseteq K_2 \supseteq \cdots$ are non-empty and compact then $\cap_{j \geq 1} K_j$ is non-empty.

4. If K is compact and $A \subseteq K$ is infinite, then A has a limit point in K.

5. The closed interval $[a, b]$ where $-\infty < a < b < \infty$ is compact.

6. If x is a limit point of a set A, there is a sequence $x_1, x_2, \ldots \in A$ converging to x.

3. Numerical Inequalities

1. Show that $1+x \leq e^x$ for every real number x. First sketch the picture. Then use calculus to rigorously establish the result. Deduce that $\log x \leq x-1$ for every $x > 0$.

2. Show that $1-x \geq e^{-x/(1-x)}$ for $0 \leq x < 1$

3. Show that $(1+u/3)^3 \geq 1+u$ for every $u \geq 0$.

4. Inequalities for $\log(1+x)$ and $\log(1-x)$ from Taylor's theorem.

   (a) Expand the function $h(v) = \log v$ in a third order Taylor series around the point $v = 1$. (Thus you will be expressing $h(1+x)$ in terms of x, $h(1)$, $h^{'}(1)$, $h^{''}(1)$, and $h^{'''}(u)$ for some u between 1 and $1+x$. Note that x may be negative.)

   (b) By examining the final term in the series, show that $\log(1+x) \geq x - x^2/2$ for $x \geq 0$.

   (c) By examining the final term in the series, show that $\log(1-x) \leq -x - x^2/2$ for $0 \leq x < 1$.

5. Show that $\log(1+x) \leq x - x^2/2 + x^3/2$ for $x \geq 0$.

6. Let $h(u) = (1+u)\log(1+u) - u$. (This function appears in Bennett's exponential inequality for sums of independent, bounded random variables.)

   (a) By considering the first few terms of the Taylor expansion of $h(\cdot)$ around zero, and bounding the remainder term, show that for every $u \geq 0$

   $$h(u) \geq \frac{u^2}{2+2u}$$

   (b) (Optional) Use calculus to establish the stronger bound that for every $u \geq 0$

   $$h(u) \geq \frac{u^2}{2+2u/3}$$

7. Show that $xy \leq 3x^2 + y^2/3$ for $x, y \geq 0$.

8. Show that $|e^a - e^b| \leq e^b\, e^{|a-b|}\, |a-b|$.

9. Let $a_1, \ldots, a_n$ be real numbers. Show that $n^{-1} \sum_{k=1}^n |a_k| \leq (n^{-1} \sum_{k=1}^n a_k^2)^{1/2}$.

10. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be numbers in the interval $[-1, 1]$. Establish the inequality

$$|a_1 \cdots a_n - b_1 \cdots b_n| \ \leq \ \sum_{i=1}^n |a_i - b_i|$$

Hint: Use induction and the fact that $a_1 a_2 - b_1 b_2 = (a_1 - b_1) a_2 + b_1 (a_2 - b_2)$.

11. Let $a_1, \ldots, a_n$ be real numbers, and let $b_1, \ldots, b_n$ be positive. Show that

$$\min_{1 \leq i \leq n} \frac{a_i}{b_i} \ \leq \ \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \ \leq \ \max_{1 \leq i \leq n} \frac{a_i}{b_i}$$

12. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be positive constants.

(a) Use Jensen's inequality to establish the Arithmetic-Geometric mean inequality

$$\frac{1}{n} \sum_{i=1}^n a_i \ \geq \ \left(\prod_{i=1}^n a_i \right)^{1/n} .$$

(b) Establish the inequality

$$(\Pi_{k=1}^n a_k)^{1/n} \ + \ (\Pi_{k=1}^n b_k)^{1/n} \ \leq \ (\Pi_{k=1}^n (a_k + b_k))^{1/n}$$

Hint: First divide the LHS by the RHS.

3. Norms and Inner Products

1. Let $||u|| = \langle u, u \rangle^{1/2}$ be the usual Euclidean norm on $\mathbb{R}^d$. Establish the following.

   (a) $||u|| \geq 0$ with equality iff $u = 0$

   (b) $||u + v||^2 = ||u||^2 + 2\langle u, v \rangle + ||v||^2$

   (c) Cauchy-Schwartz inequality $|\langle u, v \rangle| = |u^t v| \leq ||u|| \, ||v||$

   (d) $||u + v|| \leq ||u|| + ||v||$ Hint: square the left side and use Cauchy-Schwartz

   (e) $| \, ||u|| - ||v|| \, | \leq ||u - v||$ (reverse triangle inequality)

2. Show that if $u, v \in \mathbb{R}^n$ are orthogonal then $||u||_2 + ||v||_2 \leq \sqrt{2}||u + v||_2$.

3. Let $x = (x_1, \ldots, x_d)^t \in \mathbb{R}^d$ and let $||x||$ be the Euclidean (ℓ_2) norm of x. Show that for $1 \leq i \leq d$,

$$|x_i| \leq ||x|| \leq |x_1| + \cdots + |x_d|.$$

Use the inequalities to show that if $X \in \mathbb{R}^d$ is a random vector then $\mathbb{E}||X|| < \infty$ if and only if $\mathbb{E}|X_i| < \infty$ for $1 \leq i \leq d$.

4. Let x be a vector in $\mathbb{R}^d$. Show that $||x||_\infty = \lim_{p \nearrow \infty} ||x||_p$. For $0 \leq p \leq 1$ define $||x||_p = \sum_{i=1}^d |x_i|^p$. Show that $||x||_0 = \lim_{p \searrow 0} ||x||_p$.