Derivation of Key Continuous Distributions

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Convolutions and Sums of Independent R.V.

Convolutions

Definition: The convolution $h = f * g$ of two functions $f, g : \mathbb{R} \to \mathbb{R}$ is given by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-u)g(u)\,du$$

Fact: Let $X \sim f$ and $Y \sim g$ be independent. Then

1. $h = f * g$ is a density
2. $h = f * g$ is the density of $X + Y$

Cor: If f, g, h are densities then $f * g = g * f$ and $(f * g) * h = f * (g * h)$.

Review of Univariate Normal Distribution

Univariate Normal

Recall: Given $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ the $\mathcal{N}(\mu, \sigma^2)$ distribution has the (bell-shaped) density function

$$f(x) \;=\; \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \qquad -\infty < x < \infty$$

- parameters $\mu \in \mathbb{R}$ and $\sigma > 0$ fully determine density f
- special case $\mu = 0$ and $\sigma^2 = 1$ yields *standard normal*

Notation: If X has the density f above, we write $X \sim \mathcal{N}(\mu, \sigma^2)$

Univariate Normal

Basic Properties: If $X \sim \mathcal{N}(\mu, \sigma^2)$ then

- $\mathbb{E}X = \mu$ and $\mathrm{Var}(X) = \sigma^2$
- $aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$
- $X \stackrel{d}{=} \sigma Z + \mu$ where $Z \sim \mathcal{N}(0, 1)$

Fact: If $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y \sim \mathcal{N}(\eta, \tau^2)$ are independent then

$$X + Y \sim \mathcal{N}(\mu + \eta, \sigma^2 + \tau^2)$$

Fact: If $X \sim \mathcal{N}(0, 1)$ then $\mathbb{E}X^k = 1 \times 3 \times \cdot \times (2k - 1)$ if k is even, and $\mathbb{E}X^k = 0$ when k is odd

Derivation of Some Common Univariate Distributions

The Gamma Function and Stirling's Approximation

Definition: The gamma function is $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$ for $\alpha > 0$

Basic Properties of the Gamma Function

(1) $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$

(2) $\Gamma(\alpha + 1) = \alpha\, \Gamma(\alpha)$

(3) $\Gamma(n) = (n-1)!$

***Stirling's Approximation**

(1) $\Gamma(\alpha + 1) \sim \sqrt{2\pi\alpha}\, \left(\frac{\alpha}{e}\right)^{\alpha}$

(2) $n! = \sqrt{2\pi n}\, \left(\frac{n}{e}\right)^{n} \delta_n$ where $e^{1/(12n+1)} \le \delta_n \le e^{1/(12n)}$

Exponential and Double Exponential

Recall: For $\theta > 0$ the Exp(θ) distribution has density $f(x) = \theta^{-1} e^{-x/\theta}$ for $x > 0$

Fact: If $X \sim$ Exp(θ) then $\mathbb{E}X = \theta$ and $\mathrm{Var}(X) = \theta^2$

Fact: If $X, Y \sim$ Exp(θ) are independent then $X - Y$ has a double exponential distribution, written DE(θ) with density

$$f(x) = \frac{1}{2\theta} e^{-|x|/\theta} \quad -\infty < x < \infty$$

Gamma Distribution

Definition: The gamma distribution with parameters $\alpha, \beta > 0$, denoted Gam(α, β), has density

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} \quad x > 0$$

Terminology: α is called the *shape* parameter, β is called the *scale* parameter

Fact

1. If $X \sim$ Gam(α, β) then $\mathbb{E}X = \alpha\beta$ and $\mathrm{Var}(X) = \alpha\beta^2$
2. If $X \sim$ Gam(α, β) and $s > 0$ then $sX \sim$ Gam$(\alpha, s\beta)$

Fact: If $X \sim$ Gam(α_1, β) and $Y \sim$ Gam(α_2, β) are independent then the sum $X + Y \sim$ Gam$(\alpha_1 + \alpha_2, \beta)$

Beta Function

Note: From the proof of the Fact we obtain a useful identity: for each $r, s > 0$

$$B(r, s) \;:=\; \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)} \;=\; \int_0^1 (1-u)^{r-1} u^{s-1}\, du$$

The function $B(r, s)$ is called the *beta function*

Beta Distribution

Definition: For $r, s > 0$ the beta distribution, denoted Beta(r, s), has density

$$f(x) = \frac{x^{r-1}(1-x)^{s-1}}{B(r,s)} \quad 0 < x < 1$$

Note: Shape of Beta(r, s) density is flexible (see Wikipedia page)

- symmetric about $1/2$ if $r = s$
- skewed right if $1 < r < s$, skewed left if $1 < s < r$
- uniform if $r = s = 1$, unimodal if $r = s > 1$, U-shaped if $r = s < 1$

Fact: If $X \sim$ Beta(r, s) then

$$\mathbb{E}X = \frac{r}{r+s} \qquad \mathrm{Var}(X) = \frac{rs}{(r+s)^2(r+s+1)}$$

Chi-Squared Distribution

Definition: The chi-squared distribution with $k \geq 1$ degrees of freedom, written χ_k^2, is equal to Gam$(k/2, 2)$. Thus the χ_k^2 distribution has density

$$f(x) \ = \ \frac{1}{2^{k/2}\Gamma(k/2)} \, x^{k/2-1} \, e^{-x/2}$$

Useful fact: The χ_k^2 distribution can be represented as a sum of squared normal random variables: if $Z_1, \ldots, Z_k$ are iid $\mathcal{N}(0, 1)$ then $Z_1^2 + \cdots Z_k^2 \sim \chi_k^2$

Cor: If $X \sim \chi_k^2$ then $\mathbb{E}X = k$ and $\mathrm{Var}(X) = 2k$

Student's t Distributions

Definition: The t-distribution with $n \geq 1$ degrees of freedom, written t_n, is the distribution of the ratio

$$\frac{X}{\sqrt{Y/n}}$$

where $X \sim \mathcal{N}(0,1)$ and $Y \sim \chi^2_n$ are independent

Fact: The t_n distribution has density

$$f_n(x) \;=\; \frac{1}{\sqrt{\pi n}} \, \frac{\Gamma((n+1)/2)}{\Gamma(n/2)} \, \frac{1}{(1 + x^2/n)^{(n+1)/2}}$$

Student's t Distributions

Recall t_n distribution has density

$$f_n(x) \;=\; \frac{1}{\sqrt{\pi n}}\,\frac{\Gamma((n+1)/2)}{\Gamma(n/2)}\,\frac{1}{(1+x^2/n)^{(n+1)/2}}$$

Fact:

1. t_1 coincides with the Cauchy distribution
2. Mean of t_n is undefined if $n = 1$, zero if $n \geq 2$
3. Variance of t_n is undefined if $n = 1, 2$, equal to $n/(n-2)$ if $n \geq 3$
4. Density $f_n(x) \to \phi(x)$ (density of standard normal) as n tends to infinity

Stein's Lemma

Stein's Lemma

Stein's Lemma: Let $Z \sim \mathcal{N}(0,1)$ and let $f : \mathbb{R} \to \mathbb{R}$ have derivative f'. If the expectation of $f'(Z)$ is finite then

$$\mathbb{E}(Zf(Z)) = \mathbb{E}f'(Z)$$

Proof: Integration by parts.

Corollary: If $X \sim \mathcal{N}(\mu, \sigma^2)$ and the expectation of $f'(Z)$ is finite then

$$\mathbb{E}((X - \mu)f(X)) = \sigma^2 \mathbb{E}f'(X)$$

Application: Moments of the Normal Distibution

Let $X \sim \mathcal{N}(0, \sigma^2)$. We know $\mathbb{E}X = 0$ and $\mathbb{E}X^2 = \sigma^2$. What about higher moments?

Fact: If $X \sim \mathcal{N}(0, \sigma^2)$ then $\mathbb{E}X^k = 0$ when k odd and for all $k \geq 1$

$$\mathbb{E}X^{2k} = \sigma^{2k} \prod_{l=1}^{k} (2l - 1)$$