Gaussian Extreme Values

Andrew Nobel

March, 2023

Expected Maxima

MGF Bound on Expected Maxima

Task: Given rv $X_1, \ldots, X_n \in \mathbb{R}$ find a bound on $\mathbb{E}\max(X_1, \ldots, X_n)$

Gaussian Case: If $X_1, \ldots, X_n \sim \mathcal{N}(0, \sigma^2)$ then

$$\mathbb{E}\max(X_1, \ldots, X_n) \leq \sigma\sqrt{2\log n}$$

General case: If $X_1, \ldots, X_n$ satisfy $M_{X_i}(s) \leq M(s)$ for each i and all $s \geq 0$ then

$$\mathbb{E}\max(X_1, \ldots, X_n) \ \leq \ \inf_{s : M(s) \geq 1} \frac{\log n + \log M(s)}{s}$$

Note: In both results the random variables X_i need *not* be independent

Essential Supremum

Definition: The essential supremum of a random variable $X \sim F$ is given by

$$||X||_\infty \;=\; \inf\{u : \mathbb{P}(X \leq u) = 1\} \;=\; \inf\{u : F(u) = 1\}$$

- $||X||_\infty < \infty$ if and only if X is bounded above wp1
- By definition, $\mathbb{P}(X \leq ||X||_\infty - \epsilon) < 1$ for all $\epsilon > 0$
- By right continuity of F, $\mathbb{P}(X \leq ||X||_\infty) = 1$

Fact: If $X_1, X_2, \ldots$ are iid then $\max(X_1, \ldots, X_n) \to ||X||_\infty$ as n tends to infinity

More Refined Analysis: Extreme Value Theory

Setting: Let $X_1, X_2, \ldots \in \mathbb{R}$ be iid with CDF F. Interested in the limiting behavior of the maximum $M_n = \max(X_1, \ldots, X_n)$

Question: Are there scaling and centering constants $\{a_n\}$ and $\{b_n\}$ such that

$$\tilde{M}_n = a_n(M_n - b_n) \text{ has limiting CDF } G? \qquad (\star)$$

Extreme Value Theorem: If ($\star$) holds then $G(x) = G_0(ax + b)$ where a, b are constants and one of the following is true

(1) $G_0(x) = \exp(-e^{-x})$

(2) $G_0(x) = \exp(-x^{-\alpha})\,\mathbb{I}(x > 0)$ for some $\alpha > 0$

(3) $G_0(x) = \exp(-(-x)^{\alpha})\,\mathbb{I}(x \leq 0) + \mathbb{I}(x > 0)$ for some $\alpha > 0$

Preliminary Result

Fact: Let $X_1, X_2, \ldots \in \mathbb{R}$ be iid with CDF F. Let $M_n = \max(X_1, \ldots, X_n)$ and $\tau \geq 0$. For any sequence $u_1, u_2, \ldots \in \mathbb{R}$ the following are equivalent

(1) $n\,(1 - F(u_n)) \to \tau$

(2) $\mathbb{P}(M_n \leq u_n) \to e^{-\tau}$

Gaussian Tail Bound

Fact: Let $Z \sim \mathcal{N}(0,1)$ with density $\phi(x)$. For each $x > 0$ we have

$$\left(\frac{1}{x} - \frac{1}{x^3}\right) \phi(x) \ \leq \ \mathbb{P}(Z \geq x) \ \leq \ \frac{\phi(x)}{x}$$

- $\mathbb{P}(Z \geq x) = 1 - \Phi(x)$ where Φ is the CDF of Z
- Upper bound is less than $x^{-1} e^{-x^2/2} \leq e^{-x^2/2}$ when $x \geq 1$
- Result shows that $(1 - \Phi(x)) = \frac{\phi(x)}{x}(1 + O(x^{-2}))$ as $x \to \infty$
- For example $.0202 \leq \mathbb{P}(Z \geq 2) \leq .0269$ and $.0016 \leq \mathbb{P}(Z \geq 3) \leq .0022$

Maxima of Gaussian Random Variables

Basic question: Given $Z_1, Z_2, \ldots$ iid $\sim \mathcal{N}(0,1)$, interested in the limiting behavior of

$$M_n \;:=\; \max(Z_1, \ldots, Z_n)$$

Note: MGF bound shows that $\mathbb{E}M_n \leq \sqrt{2\log n}$.

First Results on Gaussian Extremes

Fact: Let $\Phi^{-1}(s)$ be the inverse CDF (percentile function) for $Z \sim \mathcal{N}(0,1)$. Then

$$\frac{\Phi^{-1}(1 - t^{-1})}{\sqrt{2 \log t}} \to 1 \text{ as } t \to \infty$$

Example: Let $z(\alpha) = \Phi^{-1}(1 - \alpha)$ be the upper α percentile of $\mathcal{N}(0,1)$. Fact shows that $z(\alpha)$ grows like $\sqrt{2 \log(1/\alpha)}$ as $\alpha \to 0$.

Fact: If $Z_1, Z_2, \ldots$ be iid $\sim \mathcal{N}(0,1)$ then

$$\frac{\mathbb{E} \max(|Z_1|, \ldots, |Z_n|)}{\sqrt{2 \log n}} \to 1 \text{ as } n \to \infty$$

Gaussian Extreme Value Theorem

Define *scaling* constants $\{a_n\}$ and *centering* constants $\{b_n\}$ as follows

$$a_n = \sqrt{2\log n} \qquad b_n = \sqrt{2\log n} - \frac{\log(4\pi\log n)}{\sqrt{8\log n}}$$

Theorem: If $Z_1, Z_2, \ldots$ iid $\sim \mathcal{N}(0,1)$ and $M_n = \max(Z_1, \ldots, Z_n)$ then for $x \in \mathbb{R}$,

$$\mathbb{P}(a_n(M_n - b_n) \leq x) \;\to\; \exp\{-e^{-x}\}$$

Note: Limiting CDF in theorem is that of $-\log U$ with $U \sim$ Exp(1)

Gaussian Extreme Value Theorem, cont.

First Corollaries

1. $M_n = b_n + O_p(1/\sqrt{\log n})$
2. $\mathbb{P}(M_n \geq \sqrt{2\log n}) \to 0$