Gaussian Comparison

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Gaussian Comparison

Given: $X \sim \mathcal{N}(0, \sigma_1^2)$ and $Y \sim \mathcal{N}(0, \sigma_2^2)$. Note that X is less "spread out" than Y if $\sigma_1^2 \leq \sigma_2^2$. In this case

- $\mathbb{P}(|X| > t) \leq \mathbb{P}(|Y| > t)$
- $\mathbb{E}|X| \leq \mathbb{E}|Y|$

Proposition: Let $X \sim \mathcal{N}_n(0, \Sigma_1)$ and $Y \sim \mathcal{N}_n(0, \Sigma_2)$. If $\Sigma_1 \leq \Sigma_2$ then for every convex set $C \subseteq \mathbb{R}^n$

$$\mathbb{P}(X \in C^c) \leq 2\mathbb{P}(Y \in C^c)$$

If $C = -C$ then the constant 2 can be improved to 1, and we have

- $\mathbb{P}(||X|| > t) \leq \mathbb{P}(||Y|| > t)$
- $\mathbb{E}||X|| \leq \mathbb{E}||Y||$

Representation of Correlated Normals

Let $(X_1, X_2)^t \sim \mathcal{N}_2$ with $\mathbb{E}X_i = 0$, $\mathrm{Var}(X_i) = 1$, and $\mathrm{Cor}(X_1, X_2) = \rho$

Goal: Explicit representation of X_1, X_2 that captures common structure

Approach: Let U_1, U_2, U be iid $\mathcal{N}(0, 1)$. Define

$$X_i' \;=\; \sqrt{\rho}\, U + \sqrt{1-\rho}\, U_i$$

Easy to see that $(X_1', X_2') \stackrel{\mathsf{d}}{=} (X_1, X_2)$. Idea

- U captures common signal in X_1, X_2
- U_i captures individual signal of X_i

Gaussian Comparison Lemma

Given: $X \sim \mathcal{N}_n(0, \Sigma_1)$ and $Y \sim \mathcal{N}_n(0, \Sigma_2)$. For $1 \leq i, j \leq n$ define

$$\Delta_{ij} \;=\; \mathbb{E}(Y_i Y_j) - \mathbb{E}(X_i X_j) \;=\; \mathrm{Cov}(Y_i, Y_j) - \mathrm{Cov}(X_i X_j) \;=\; (\Sigma_1 - \Sigma_2)_{ij}$$

Lemma: Let $G : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable. Define

$$G_i = \partial G / \partial x_i \quad G_{ij} = \partial^2 G / \partial x_i \partial x_j$$

Under suitable regularity conditions on G,

$$\mathbb{E}G(Y) - \mathbb{E}G(X) \;=\; \frac{1}{2} \sum_{i,j=1}^{n} \Delta_{ij} \int_0^1 \mathbb{E}G_{ij}(X(t))\, dt$$

where $X(t) \sim \mathcal{N}_n(0, (1-t)\Sigma_1 + t\Sigma_2)$

Gaussian Comparison Lemma, cont.

Corollary: Let X, Y, Δ_{ij}, and G be as above. If

- $G_{ij}(\cdot) \geq 0$ when $\Delta_{ij} > 0$
- $G_{ij}(\cdot) \leq 0$ when $\Delta_{ij} < 0$

then $\mathbb{E}G(Y) \geq \mathbb{E}G(X)$

Slepian's Lemma

Lemma: Let X, Y be $\mathcal{N}_n$ with $\mathbb{E}X = \mathbb{E}Y = 0$ and $\mathbb{E}X_i^2 = \mathbb{E}Y_i^2$ for all i. If

$$\mathbb{E}X_iX_j \ \le \ \mathbb{E}Y_iY_j \text{ for all } i \neq j$$

then for all $u_1, \ldots, u_n \in \mathbb{R}$

$$\mathbb{P}(X_i \le u_i \text{ for } 1 \le i \le n) \ \le \ \mathbb{P}(Y_i \le u_i \text{ for } 1 \le i \le n)$$

Corollaries

1. $\mathbb{P}(\max Y_i > u) \le \mathbb{P}(\max X_i > u)$ for all u
2. $\mathbb{E}(\max_i Y_i) \le \mathbb{E}(\max_i X_i)$