Theoretical Statistics, STOR 655

Asymptotic Normality of Method of Moments Estimates

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Inverse Function Theorem

Theorem: Let $\phi : \mathbb{R}^k \to \mathbb{R}^k$ be such that

a. $D\phi(\cdot)$ is defined and continuous in a neighborhood of $\theta_0 \in \mathbb{R}^k$

b. $D\phi(\theta_0) \in \mathbb{R}^{k \times k}$ is invertible

Then there exist open $U, V \subseteq \mathbb{R}^k$ with $\theta_0 \in U$ and $\phi(\theta_0) \in V$ such that

1. $\phi^{-1} : V \to U$ is well defined and continuous
2. ϕ^{-1} is differentiable at $v_0 = \phi(\theta_0)$
3. $D\phi^{-1}(v_0) = (D\phi(\theta_0))^{-1}$

Inference Setting

Setting

- Statistical model: Family $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ of distributions on a measurable space $(\mathcal{X}, \mathcal{S})$ indexed by a parameter set $\Theta \subseteq \mathbb{R}^k$
- Observations $X_1, X_2, \ldots \in \mathcal{X}$ iid with $X_i \sim P_{\theta_0}$. The parameter $\theta_0 \in \Theta$ is unknown
- Goal: Point estimate of θ_0 based on $X_1, \ldots, X_n$

Notation: For $f : \mathcal{X} \to \mathbb{R}^k$ define

- $P_\theta f = \int_{\mathcal{X}} f(x) dP_\theta$
- $P_n f = n^{-1} \sum_{i=1}^n f(X_i)$

Method of Moments

Preliminaries:

- Fix test functions $f_1, \ldots, f_k : \mathcal{X} \to \mathbb{R}$ such that $P_\theta |f_j| < \infty$ for all $\theta \in \Theta$
- Let $f = (f_1, \ldots, f_k)^t$

Definition: Method of moments estimator $\hat{\theta}_n^{\text{MoM}}(X_1^n)$ is any any solution $\theta \in \Theta$ of the equation $P_n f = P_\theta f$

Idea: $\hat{\theta}_n^{\text{MoM}}(X_1^n)$ is any parameter θ such that the sample means of the test functions $f_1, \ldots, f_k$ are equal to their population means under P_θ

Consistency and Asymptotic Normality of MoM

Theorem: Define $\phi : \Theta \to \mathbb{R}^k$ by $\phi(\theta) = P_\theta f = \int_{\mathcal{X}} (f_1, \ldots, f_k)^t dP_\theta$. Assume

a. ϕ is 1:1 on Θ and $\theta_0 \in \Theta^o$

b. $D\phi(\cdot)$ is defined continuous at θ_0 and $D_0 := D\phi(\theta_0)$ is invertible

c. $P_{\theta_0} ||f||^2 < \infty$

Then

1. $\hat{\theta}_n^{\text{MoM}}$ exists with probability tending to one
2. $\hat{\theta}_n^{\text{MoM}} \to \theta_0$ with probability one
3. $n^{1/2}(\hat{\theta}_n^{\text{MoM}} - \theta_0) \Rightarrow \mathcal{N}_k(0, D_0^{-1} \Sigma (D_0^{-1})^t)$ where

$$\Sigma = P_{\theta_0}(f f^t) - (P_{\theta_0} f)(P_{\theta_0} f)^t$$

Example: Beta Distribution

Given: Observations $X_1, \ldots, X_n \in \mathbb{R}$ iid $\sim$ Beta(α, β) with density

$$f_{\alpha,\beta}(x) \;=\; \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

MoM estimate of (α, β) with $f(x) = (x, x^2)^t$ is the solution $(\hat{\alpha}, \hat{\beta})$ of

$$\overline{X}_n \;=\; \mathbb{E}_{\alpha,\beta} X \;=\; \frac{\alpha}{\alpha+\beta}$$

$$\overline{X}_n^2 \;=\; \mathbb{E}_{\alpha,\beta} X^2 \;=\; \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}$$