Theoretical Statistics, STOR 655

Likelihood Ratio Test

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Likelihood Ratio Test

Family $\mathcal{P} = \{f(x|\theta) : \theta \in \Theta\}$ of densities on $(\mathcal{X}, \mathcal{A})$, base measure ν

- Suppose $\Theta \subseteq \mathbb{R}^p$ partitioned as $\Theta = \Theta_0 \cup \Theta_1$

Observations $X_1, X_2, \ldots \in \mathcal{X}$ iid with $X_i \sim f(x|\theta')$

- Wish to test $\mathrm{H}_0 : \theta' \in \Theta_0$ vs $\mathrm{H}_1 : \theta' \in \Theta_1$ based on $X_1, \ldots, X_n$

Recall: Let $L_n(\theta)$ be the likelihood function of $X_1, \ldots, X_n$. The likelihood ratio test statistic is

$$\lambda_n = \frac{\sup_{\theta \in \Theta_0} L_n(\theta)}{\sup_{\theta \in \Theta} L_n(\theta)}$$

The likelihood ratio test rejects H_0 if λ_n is small

Likelihood Ratio Test

Note: Likelihood ratio test statistic can be written as

$$\lambda_n = \frac{L_n(\theta_n^*)}{L_n(\hat{\theta}_n)}$$

where

$$\theta_n^* = \underset{\theta \in \Theta_0}{\operatorname{argmax}} L_n(\theta) \qquad \hat{\theta}_n = \underset{\theta \in \Theta}{\operatorname{argmax}} L_n(\theta)$$

are the restricted and unrestricted MLEs of the true parameter θ'

Goal: Identify asymptotic distribution of λ_n for appropriate nulls Θ_0

Asymptotic Distribution of Likelihood Ratio Test Statistic

Assumption: For some $1 \leq k \leq p$ we have

$$\Theta_0 \; = \; \{\theta \in \Theta : \theta_1 = \cdots = \theta_k = 0\} \; \neq \; \emptyset$$

Equivalently, we may write $\mathrm{H}_0 : \theta_1 = \cdots = \theta_k = 0$.

Note: k is the number of restrictions on θ under the null.

Theorem: Let Θ_0 be as above, and suppose the conditions for the asymptotic normality of the MLE hold. If $\theta' \in \Theta_0$ then

$$-2 \log \lambda_n \; \Rightarrow \; \chi^2_k$$