Symmetrization and Contraction

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Ingredients

1. Independent stochastic processes $Z_1, \ldots, Z_m$

- $Z_i = \{Z_i(t) : t \in T\}$ have common index set T
- $\mathbb{E}Z_i(t)$ well defined for all $1 \leq i \leq m$ and $t \in T$

2. Independent Rademacher (sign) variables $\varepsilon_1, \ldots, \varepsilon_m$

- $\mathbb{P}(\varepsilon_i = +1) = \mathbb{P}(\varepsilon_i = -1) = 1/2$
- Sign variables $\{\varepsilon_i\}$ independent of processes $\{Z_i\}$

Symmetrization and Contraction

Thm (Symmetrization)

$$\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^{m} (Z_i(t) - \mathbb{E} Z_i(t)) \right| \;\leq\; 2\, \mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^{m} \varepsilon_i \, Z_i(t) \right|$$

Thm (Contraction)

$$\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^{m} \varepsilon_i \, |Z_i(t)| \right| \;\leq\; 2\, \mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^{m} \varepsilon_i \, Z_i(t) \right|$$

Key Lemma

Lemma: If $L \subseteq \mathbb{R}^n$ is bounded and $V_1, \ldots, V_m \sim \mathcal{N}_n(0, I)$ are iid then

$$\mathbb{E} \sup_{u \in L} \left| \frac{1}{m} \sum_{i=1}^{m} |\langle u, V_i \rangle| - \sqrt{\frac{2}{\pi}} ||u||_2 \right| \;\leq\; \frac{4}{\sqrt{m}} \, \mathbb{E} \sup_{u \in L} |\langle u, V \rangle|$$