GAUSSIAN COMPARISON LEMMA

1. Gaussian Comparision Lemma

Lemma 1.1. *Let $G : \mathbb{R}^n \to \mathbb{R}$ be a bounded, twice continuously differentiable function with bounded derivatives*

$$G_i(x) = \frac{\partial G(x)}{\partial x_i} \quad 1 \leqslant i \leqslant n \qquad \textit{and} \qquad G_{ij} = \frac{\partial G(x)}{\partial x_i \partial x_j} \quad 1 \leqslant i, j \leqslant n.$$

If $\mathbf{X} \sim \mathcal{N}_n(0, \Sigma_X)$ and $\mathbf{Y} \sim \mathcal{N}_n(0, \Sigma_Y)$ are normal random vectors then

$$\mathbb{E}\, G(\mathbf{Y}) - \mathbb{E}\, G(\mathbf{X}) = \frac{1}{2} \sum_{i,j=1}^{n} \Delta_{ij} \int_0^1 \mathbb{E}\, G_{ij}(\mathbf{X}^t)\, dt$$

where $\Delta_{ij} = \mathbb{E}\, Y_i Y_j - \mathbb{E}\, X_i X_j = (\Sigma_Y - \Sigma_X)_{ij}$ and $\mathbf{X}^t \sim \mathcal{N}_n(0, \Sigma_t)$ with $\Sigma_t := (1-t)\, \Sigma_X + t\, \Sigma_Y$.

Proof: Assume without loss of generality that $\mathbf{X}$ and $\mathbf{Y}$ are independent. For each $t \in [0, 1]$ define the random vector

$$\mathbf{X}^t = (1-t)^{1/2}\, \mathbf{X} + t^{1/2}\, \mathbf{Y}$$

and the associated function $\varphi(t) = \mathbb{E}\, G(\mathbf{X}^t)$. Note that $\mathbf{X}^0 = \mathbf{X}$, $\mathbf{X}^1 = \mathbf{Y}$, and that $\mathbf{X}^t \sim \mathcal{N}_n(0, \Sigma_t)$, where Σ_t is defined as in the statement of the lemma. Thus

$$\mathbb{E}\, G(\mathbf{Y}) - \mathbb{E}\, G(\mathbf{X}) \;=\; \varphi(1) - \varphi(0) = \int_0^1 \varphi'(t)\, dt,$$

and it suffices to show that for each $t \in (0, 1)$

phipr0 (1.1)
$$\varphi'(t) = \frac{1}{2} \sum_{i,j=1}^{n} \Delta_{ij}\, \mathbb{E}\, G_{ij}(\mathbf{X}^t). \tag{1.1}$$

To this end, fix $t \in (0, 1)$ and note that $\mathbf{X}^t$ is distributed as $\Sigma_t^{1/2} \mathbf{Z}$ where $\mathbf{Z} \sim \mathcal{N}(0, I)$ is a standard normal random vector with independent components. To simplify notation, let $A_t := \Sigma_t^{1/2}$. It follows from our regularity assumptions and the chain rule that

$$\varphi'(t) \;=\; \frac{d}{dt}\, \mathbb{E}\, G(A_t\, \mathbf{Z}) \;=\; \mathbb{E} \left[\frac{d}{dt} G(A_t\, \mathbf{Z}) \right] \;=\; \mathbb{E} \left[\sum_{i=1}^{n} G_i(A_t\, \mathbf{Z})\, \frac{d}{dt}\, (A_t\, \mathbf{Z})_i \right]$$

phipr1 (1.2)
$$\;=\; \sum_{i,j=1}^{n} (A_t')_{ij}\, \mathbb{E}\, (Z_j\, G_i(A_t\, \mathbf{Z}))\,, \tag{1.2}$$

where A_t' denotes the entry-by-entry derivative of the matrix A_t. Fix i, j for the moment and define the function

$$H_{ij}(s) \;:=\; \mathbb{E}\, G_i(A_t\, \mathbf{Z}_s) \quad \text{where} \quad \mathbf{Z}_s \;:=\; (Z_1, \cdots, Z_{j-1}, s, Z_{j+1}, \cdots, Z_n).$$

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It follows from a simple conditioning argument and Gaussian integration by parts that

GIP (1.3)
$$\mathbb{E}\,[Z_j\, G_i(A_t\mathbf{Z})] \;=\; \mathbb{E}\,[Z_j\, H_{ij}(Z_j)] \;=\; \mathbb{E}\, H_{ij}'(Z_j). \tag{1.3}$$

By another application of the chain rule,

$$\begin{aligned} H_{ij}'(s) \;&=\; \mathbb{E}\left[\frac{d}{ds} G_i(A_t\,\mathbf{Z}_s)\right] \;=\; \sum_{k=1}^{n} \mathbb{E}\left[G_{ik}(A_t\,\mathbf{Z}_s)\,\frac{d}{dt}(A_t\,\mathbf{Z}_s)_k\right] \\ &=\; \sum_{k=1}^{n} (A_t)_{jk}\,\mathbb{E}\, G_{ik}(A_t\,\mathbf{Z}_s). \end{aligned}$$

Thus, as $Z_1, \ldots, Z_n$ are independent,

$$\mathbb{E}\, H_{ij}'(Z_j) = \sum_{k=1}^{n} (A_t)_{jk}\,\mathbb{E}\, G_{ik}(A_t\,\mathbf{Z}).$$

Combining this last equation with (1.2), we find that

phipr2 (1.4)
$$\begin{aligned} \varphi'(t) \;&=\; \sum_{i,k=1}^{n} \mathbb{E}\, G_{ik}(A_t\,\mathbf{Z}) \cdot \sum_{j=1}^{n} (A_t')_{ij}(A_t)_{jk} \\ &=\; \sum_{i,k=1}^{n} \mathbb{E}\, G_{ik}(\mathbf{X}^t) \cdot (A_t'\,A_t)_{ik}. \end{aligned} \tag{1.4}$$

Recalling that $A_t = \Sigma_t^{1/2}$, it is easy to see that $(A_t^2)_{ik}' = (\Sigma_t)_{ik}' = \Delta_{ik}$. Furthermore, as A_t and A_t' are symmetric,

atsq (1.5)
$$(A_t^2)' \;=\; A_t'\,A_t \,+\, A_t\,A_t' \;=\; A_t'\,A_t \,+\, (A_t'\,A_t)^T. \tag{1.5}$$

Fix $1 \leqslant i < k \leqslant n$. Continuity of the second partial derivatives ensures that $G_{ik} = G_{ki}$, and therefore

$$\begin{aligned} &\mathbb{E}\, G_{ik}(\mathbf{X}^t) \cdot (A_t'\,A_t)_{ik} \,+\, \mathbb{E}\, G_{ki}(\mathbf{X}^t) \cdot (A_t'\,A_t)_{ki} \\ &\quad=\; \mathbb{E}\, G_{ik}(\mathbf{X}^t)\,\big((A_t'A_t)_{ik} \,+\, (A_t'A_t)_{ki}\big) \\ &\quad=\; \mathbb{E}\, G_{ik}(\mathbf{X}^t)\,(A_t^2)_{ik}' \;=\; \mathbb{E}\, G_{ik}(\mathbf{X}^t)\,\Delta_{ik}, \end{aligned}$$

where the penultimate equality follows from (1.5). A similar argument shows that $(A_t'\,A_t)_{ii} = \Delta_{ii}/2$. Thus (1.1) follows from (1.2), and the proof is complete.

1.1. **Further Reading: Gaussian Tail Bounds.** Let $\bar{\Phi}(x) = 1 - \Phi(x)$ where $\Phi(x)$ is the cumulative distribution function of the standard Gaussian distribution. Recall that for $x > 0$,

eq:g1
$$\bar{\Phi}(x) \leqslant \frac{1}{\sqrt{2\pi}x} e^{-x^2/2}. \tag{1.6}$$

The proof of Theorem **??** requires an inequality for the probability that two correlated Gaussian random variables each exceeds a common threshold.

:biv-norm-tail **Lemma 1.2.** *Let (Z, Z_ρ) be jointly Gaussian random variables with mean 0, variance 1, and correlation $\mathbb{E}(ZZ_\rho) = \rho \in (-1, 1)$. Then for any $u > 0$,*

eq:zzr0
$$\mathbb{P}(Z > u, Z_\rho > u) \;\leqslant\; \frac{(1+\rho)^2}{2\pi u^2 \sqrt{1-\rho^2}} \exp\left(-u^2/(1+\rho)\right). \tag{1.7}$$

Proof of Lemma 1.2. Fix $u > 0$. When $\rho \geqslant 0$ the proof follows from known inequalities in the literature (see R. Willink, Bounds on the bivariate normal distribution function, Comm. Statist. Theory Methods 33 (2004), pp.2281-2297). Here we consider the case $\rho < 0$. Note that we may write $Z_\rho = \rho Z + \sqrt{1-\rho^2}Z'$, where Z' is a standard Gaussian random variable independent of Z. By conditioning on the value of Z, it is easy to see that

eq:zzr1
$$\mathbb{P}(Z > u, Z_\rho > u) \;=\; \int_u^\infty \bar{\Phi}\left(g(t)\right) \phi(t)\, dt \quad \text{where} \quad g(t) \;=\; \frac{u - \rho t}{\sqrt{1-\rho^2}}. \tag{1.8}$$

Now define

$$\eta = \sqrt{\frac{1-\rho}{1+\rho}} \;\text{ and }\; h(x) = e^{x^2/2}\, \bar{\Phi}(x).$$

As $h'(x) = x\, e^{x^2/2}\, \bar{\Phi}(x) - 1/\sqrt{2\pi}$, inequality (1.6) implies that $h(x)$ is decreasing for $x > 0$. It follows from equation (1.8) that

eq:zzr2
$$\begin{aligned} \mathbb{P}(Z > u, Z_\rho > u) \;&=\; \int_u^\infty e^{-g(t)^2/2}\, h(g(t))\, \phi(t)\, dt \\ &\leqslant\; h(g(u)) \int_u^\infty e^{-g(t)^2/2}\, \phi(t)\, dt \\ &=\; h(\eta u) \int_u^\infty e^{-g(t)^2/2}\, \phi(t)\, dt, \end{aligned} \tag{1.9}$$

where in the last step we have used the fact that $g(u) = \eta u$. Routine algebra and a change of variables establishes that

$$\begin{aligned} \int_u^\infty e^{-g(t)^2/2}\, \phi(t)\, dt \;&=\; e^{-u^2/2} \int_u^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(t-\rho u)^2}{2(1-\rho^2)}\right) dt \\ &=\; \sqrt{1-\rho^2}\; e^{-u^2/2} \bar{\Phi}(\eta u). \end{aligned} \tag{1.10}$$

Combining (1.9), (1.11), and inequality (1.6) yields the bound (1.7), as desired. ■