STOR 565 Homework: Order and Convexity

1. Let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ be two sequences of real numbers.

   a. Show that $\min\{a_i\} + \min\{b_i\} \;\leq\; \min\{a_i + b_i\} \;\leq\; \min\{a_i\} + \max\{b_i\}$.

   Hints: For the first inequality, note that the leftmost term is less than or equal to $a_j + b_j$ for every j. For the second inequality, note that the middle term is less than or equal to $a_j + b_j$ where $a_j = \min\{a_i\}$.

   b. As clearly as you can, provide an English language explanation of the inequalities above.

   c. Following the arguments from the lecture, show that $\max\{-b_i\} = -\min\{b_i\}$.

   d. Use the results above to show that

$$\min\{a_i\} - \max\{b_i\} \;\leq\; \min\{a_i - b_i\} \;\leq\; \min\{a_i\} - \min\{b_i\}.$$

2. In each case below find $\min_{x \in \mathcal{X}} f(x)$, $\operatorname{argmin}_{x \in \mathcal{X}} f(x)$, $\max_{x \in \mathcal{X}} f(x)$, and $\operatorname{argmax}_{x \in \mathcal{X}} f(x)$. Indicate when the min or the max do not exist. It may help to sketch the functions.

   a. $f(x) = \sin x$ with $\mathcal{X} = [0, 2\pi]$ and $\mathcal{X} = [0, \pi]$

   b. $f(x) = x^2$ with $\mathcal{X} = [-2, 2]$, $\mathcal{X} = (-2, 2]$, $\mathcal{X} = (-2, 2)$

   c. $f(x) = \min(x, 1)$ with $\mathcal{X} = [0, 2]$ and $\mathcal{X} = (-2, 2]$

3. Define what it means for a set $C \subseteq \mathbb{R}^d$ to be convex. Let $w \in \mathbb{R}^d$ be a vector and $b \in \mathbb{R}$ a constant. Show that $C = \{x : w^t x \geq b\}$ and $D = \{x : w^t x = b\}$ are convex subsets of $\mathbb{R}^d$.

4. Let $C_1, \ldots, C_n \subseteq \mathbb{R}^d$ be convex. Show that the intersection $\cap_{i=1}^n C_i$ is convex.

5. (Operations on convex functions that produce new convex functions) Let $C \subseteq \mathbb{R}^d$ be a convex set and let $f_1, \ldots, f_n : C \to \mathbb{R}$ be convex functions. Use the definition of convexity to establish the following.

   a. If $a_1, \ldots, a_n$ are non-negative then $g(x) = \sum_{i=1}^n a_i \, f_i(x)$ is convex on C.

   b. The function $g(x) = \max_{1 \leq i \leq n} f_i(x)$ is convex on C.

c. If $h : \mathbb{R} \to \mathbb{R}$ is convex and increasing then $g(x) = h(f(x))$ is convex on C. (Recall that h is increasing if $u \leq v$ implies $h(u) \leq h(v)$).

6. Define what it means for a function to be strictly convex. Define the notion of a global minima. Repeat the argument from class showing that the global minima of a strictly convex function is necessarily unique.

7. Let $h_\alpha : \mathbb{R} \to [0, \infty)$ be defined by $h_\alpha(x) = |x|^\alpha$ where $\alpha > 0$ is fixed. Sketch $h_\alpha(x)$ for $\alpha = 1/2, 1, 2$. For which values of α is $h_\alpha(x)$ convex? Justify your answer.

8. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. For $\gamma \in \mathbb{R}$ the γ-level set of f is defined to be the set of points x where $f(x)$ is less than or equal to γ. Formally,

$$L_\gamma(f) \;=\; \{x : f(x) \leq \gamma\}$$

a. Draw some level sets for the convex functions $f(x) = x^2$ and $f(x) = e^{-x}$. Note that $L_\gamma(f)$ may be empty.

b. Show that for each γ the level set $L_\gamma(f)$ is convex. Hint: If $L_\gamma(f)$ is empty then it is trivially convex. Otherwise, use the definition of a convex set.

9. Let $U_1, \ldots, U_m$ be random variables. Find an inequality relating $\mathbb{E}(\min_{1 \leq j \leq m} U_j)$ and $\min_{1 \leq j \leq m} \mathbb{E} U_j$. Hint: Begin by noting that $\min_{1 \leq j \leq m} U_j \leq U_k$ for each k.

10. Let $f_1, \ldots, f_k : \mathbb{R}^p \to \mathbb{R}$ be convex functions.

a. Show that for each number t the set $L_t = \{x : \sum_{j=1}^k f_j(x) \leq t\}$ is convex. Hint: Use results from the previous homework.

b. Show that for each t the sets $\{\beta \in \mathbb{R}^p : \sum_{j=1}^p \beta_j^2 \leq t\}$ and $\{\beta \in \mathbb{R}^p : \sum_{j=1}^p |\beta_j| \leq t\}$ are convex.

11. Show that the Lagrange dual function, defined by

$$\tilde{L}(\lambda) \;=\; \min_{w,b} \, L(w, b, \lambda)$$

is concave. Hint: Argue that the dual function is the minimum of linear (hence concave) functions, and is therefore concave. The SVM dual problem is given by the program

$$\max \tilde{L}(\lambda) \quad \text{s.t.} \quad \sum_{i=1}^n \lambda_i \, y_i = 0 \;\text{ and }\; \lambda_1, \ldots, \lambda_n \geq 0$$

Carefully define the constraint set for λ in this problem and argue that this set is convex. (Note that there are $n+1$ constraints.) Thus the dual problem seeks to maximize a concave function over a convex set.

12. Let $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be any real valued function. Show that

$$\max_{x \in \mathcal{X}} \min_{y \in \mathcal{Y}} f(x, y) \;\leq\; \min_{y \in \mathcal{Y}} \max_{x \in \mathcal{X}} f(x, y)$$

This inequality shows that the value d^* of the SVM dual problem is less than or equal to the value p^* of the SVM primal problem.

13. Show that the following functions $f, g, h : [0, 1] \to \mathbb{R}$ used to define impurity measures for growing trees are concave.

a. $m(p) = \min(p, 1 - p)$

b. $g(p) = p(1 - p)$

c. $h(p) = -p \log p - (1 - p) \log(1 - p)$, with the convention that $0 \log 0 = 0$

Which of these functions is strictly concave?