Order, Minima, and Maxima

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Multiplication and Addition

Recall: For any numbers a, b

(1) If $a, b \geq 0$ or $a, b \leq 0$ then $ab \geq 0$

(2) If $a \geq 0$ and $b \leq 0$ or vice-versa then $ab \leq 0$

(3) If $a, b \geq 0$ then $a + b \geq 0$

(4) If $a, b \leq 0$ then $a + b \leq 0$.

Note: (1)-(4) continue to hold if we replace $\leq$ and $\geq$ by $<$ and $>$, respectively

The Usual Order Relation

Definition: For $a, b \in \mathbb{R}$ write $a \leq b$ if $(b - a) \geq 0$

Basic Properties

1. If $a \leq b$ and $b \leq a$ then $a = b$
2. If $a \leq b$ then $-b \leq -a$
3. If $a \leq b$ and $c \leq d$ then $a + c \leq b + d$
4. If $0 \leq a \leq b$ and $0 \leq c \leq d$ then $ac \leq bd$

Note: (2)-(4) continue to hold if we replace $\leq$ by $<$

Maxima and Minima of Finite Sequences

Definition: Let $a_1, \ldots, a_n \in \mathbb{R}$

- $\max\{a_1, \ldots, a_n\}$ is any element a_j such that $a_i \leq a_j$ for $i = 1, \ldots, n$
- $\min\{a_1, \ldots, a_n\}$ is any element a_j such that $a_i \geq a_j$ for $i = 1, \ldots, n$

Notation

- $\max\{a_1, \ldots, a_n\}$ sometimes written as $\max_{1 \leq i \leq n} a_i$ or simply $\max_i a_i$
- $\min\{a_1, \ldots, a_n\}$ sometimes written as $\min_{1 \leq i \leq n} a_i$ or simply $\min_i a_i$

Maxima and Minima, cont.

Basic Properties: Let $a_1, \ldots, a_n \in \mathbb{R}$ and $b_1, \ldots, b_n \in \mathbb{R}$ be finite sequences

1. If $a_i \leq b_i$ for each i, then $\max_i a_i \leq \max_i b_i$ and $\min_i a_i \leq \min_i b_i$
2. $\min_i a_i \leq a_j \leq \max_i a_i$ for $j = 1, \ldots, n$
3. $-\min_i a_i = \max_i(-a_i)$ and $-\max_i a_i = \min_i(-a_i)$
4. If b and $c \geq 0$ are constants then $c \max_i a_i + b = \max_i(c\, a_i + b)$
5. $\max_i(a_i + b_i) \leq \max_i a_i + \max_i b_i$
6. $\min_i(a_i + b_i) \geq \min_i a_i + \min_i b_i$
7. $\max_i a_i - \max_i b_i \leq \max_i |a_i - b_i|$

Order Relations for Maxima and Minima of Functions

Fact: Let $f, g : \mathcal{X} \to \mathbb{R}$ be functions.

(1) $\min_{x\in\mathcal{X}} f(x) \;\leq\; f(x_0) \;\leq\; \max_{x\in\mathcal{X}} f(x)$ for every $x_0 \in \mathcal{X}$

(2) $-\max_{x\in\mathcal{X}} f(x) \;=\; \min_{x\in\mathcal{X}}(-f(x))$

(3) $\max_{x\in\mathcal{X}}\{f(x) + g(x)\} \;\leq\; \max_{x\in\mathcal{X}} f(x) \;+\; \max_{x\in\mathcal{X}} g(x)$

(4) If $\mathcal{X}_0 \subseteq \mathcal{X}$ then $\max_{x\in\mathcal{X}_0} f(x) \;\leq\; \max_{x\in\mathcal{X}} f(x)$

Fact: If $h : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is any function, then

$$\max_{x\in\mathcal{X}} \min_{y\in\mathcal{Y}} h(x, y) \;\leq\; \min_{y\in\mathcal{Y}} \max_{x\in\mathcal{X}} h(x, y)$$

Minimax theorems, which give conditions under which the reverse inequality holds, are important in machine learning and game theory

Argmax and Argmin

Definition: The *argmax* of a function $f : \mathcal{X} \to \mathbb{R}$ is the set of points $x \in \mathcal{X}$ where $f(x)$ is maximized. Formally,

$$\begin{aligned}\operatorname*{argmax}_{x \in \mathcal{X}} f(x) &= \{x \in \mathcal{X} : f(x) \geq f(y) \text{ for all } y \in \mathcal{X}\} \\ &= \left\{x \in \mathcal{X} : f(x) = \max_{y \in \mathcal{X}} f(y)\right\}\end{aligned}$$

Similarly, the *argmin* of f is the set of points $x \in \mathcal{X}$ where $f(x)$ is minimized.

$$\begin{aligned}\operatorname*{argmin}_{x \in \mathcal{X}} f(x) &= \{x \in \mathcal{X} : f(x) \leq f(y) \text{ for all } y \in \mathcal{X}\} \\ &= \left\{x \in \mathcal{X} : f(x) = \min_{y \in \mathcal{X}} f(y)\right\}\end{aligned}$$

Argmax and Argmin, cont.

Note that $\operatorname{argmax}_{x \in \mathcal{X}} f(x)$ is a subset of $\mathcal{X}$

- $\max_{x \in \mathcal{X}} f(x)$ is the maximum value of $f(x)$ if this exists
- $\operatorname{argmax}_{x \in \mathcal{X}} f(x)$ is the set of arguments x achieving the maximum value
- $\operatorname{argmax}_{x \in \mathcal{X}} f(x)$ is non-empty iff $\max_{x \in \mathcal{X}} f(x)$ defined

Note that $\operatorname{argmin}_{x \in \mathcal{X}} f(x)$ is a subset of $\mathcal{X}$

- $\min_{x \in \mathcal{X}} f(x)$ is the minimum value of $f(x)$ if this exists
- $\operatorname{argmin}_{x \in \mathcal{X}} f(x)$ is the set of arguments x achieving the minimum value
- $\operatorname{argmin}_{x \in \mathcal{X}} f(x)$ is non-empty iff $\min_{x \in \mathcal{X}} f(x)$ defined

Examples

Ex: $f(x) = x^3$ with $\mathcal{X} = [-2, 2]$, $\mathcal{X} = (-2, 2]$,

Ex: $f(x) = |x|$ with $\mathcal{X} = [-1, 1]$, $\mathcal{X} = \mathbb{R}$

Ex: $f(x) = \min\{1, x\}$ with $\mathcal{X} = \mathbb{R}$

Ex: $f(x) = 1/(1 + e^{-x})$ with $\mathcal{X} = [0, \infty)$ and $\mathcal{X} = \mathbb{R}$