STOR 565 Homework: Calculus

1. By graphing the functions f(x) = 1+x and $g(x) = e^x$, argue informally that $1+x \le e^x$ for every number x, and find one value of x where equality holds. Deduce from this inequality that $\log y \le y - 1$ for every y > 0.

2. Let $x = x_1, \ldots, x_n$ be a univariate sample of n numbers. It is a standard, and important, fact that the quantity $h(a) = \sum (x_i - a)^2$ is minimized when (and only when) a is the sample mean $m(x) = n^{-1} \sum_{i=1}^n x_i$. Here we show this in two different ways.

- a. Take a derivative of h to find the number a that minimizes or maximizes the function h, and then take another derivative to show that the number you found minimizes the function.
- b. Consider the expression for h. Add and subtract m(x) inside the parentheses, expand the square, and take the sum of these terms. Note that one of the sums is zero, and one of the terms does not depend on a. Use this to show that the sample mean minimizes h(a).
- c. Use what you've shown above to find the following

$$\operatorname{argmin}_{a \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} (x_i - a)^2 \quad \text{and} \quad \min_{a \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} (x_i - a)^2$$

- 3. (Inequalities from Calculus) Use calculus to establish the following inequalities.
 - a. $(1 + u/3)^3 \ge 1 + u$ for every $u \ge 0$
 - b. $x + x^{-1} \ge 2$ for $x \ge 1$
 - c. $\log(1+x) \ge x x^2/2$ for $x \ge 0$. Note that this inequality requires taking a second derivative to show that the first derivative is increasing.

4. Use the second derivative condition to establish whether the following functions are convex or concave. In each case, sketch the function.

- a. The function $f(x) = e^x$ on $(-\infty, \infty)$.
- b. The function $f(x) = \sqrt{x}$ on $(0, \infty)$.

- c. The function f(x) = 1/x on $(0, \infty)$.
- d. The function $f(x) = \log x$ on $(0, \infty)$.

Now let X > 0 be a positive random variable. Write out the conclusion of Jensen's inequality for each of the functions above.

- 5. Define the function $f(x) = x \log x$ for $x \in (0, \infty)$
 - a. Sketch the function f(x) and show that it is convex.
 - b. Find the minimum and argmin of f(x).
 - b. Let X > 0 be a random variable. What can you say about the relationship between $\mathbb{E}(X \log X)$ and $\mathbb{E}X \log \mathbb{E}X$?
- 6. Show that for each number $u \in \mathbb{R}$ we have

$$\min(u, 1-u) = u \mathbb{I}(u < 1/2) + (1-u) \mathbb{I}(u \ge 1/2)$$

Hint: Consider separately the cases u < 1/2 and $u \ge 1/2$.

7. Find the gradient and Hessian of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x) = x_1^2 x_2 + 3x_1 - 5x_2 + 1$$

- 8. Let $f : \mathbb{R}^d \to \mathbb{R}$ be defined by $f(\mathbf{x}) = \mathbf{x}^t \mathbf{A} \mathbf{x}$ where $\mathbf{A} \in \mathbb{R}^{d \times d}$ is symmetric.
 - a. Show that the gradient of f is given by $\nabla f(\mathbf{x}) = 2\mathbf{A}\mathbf{x}$.
 - b. Show that the Hessian of f is given by $\nabla^2 f(\mathbf{x}) = 2\mathbf{A}$.
- 9. Use calculus to show that for $u \in (0, 1)$

$$\frac{u^2}{2(1-u)} \ge -\log(1-u) - u$$