STOR 565 Homework

1. Let P be a probability measure on a set $\mathcal{X}$. Recall that if A and B are subsets of $\mathcal{X}$ and $P(B) > 0$, then the conditional probability of A given B is defined by

$$P(A \,|\, B) \;=\; \frac{P(A \cap B)}{P(B)}$$

Show the following.

a. If A and B are disjoint then $P(A \cup B \,|\, C) = P(A \,|\, C) + P(B \,|\, C)$

b. $P(A^c \,|\, B) = 1 - P(A \,|\, B)$

c. If $A \subseteq B$ then $P(A \,|\, C) \leq P(B \,|\, C)$

2. Let $\mathcal{X}$ be a set and let A, B be subsets of $\mathcal{X}$. Recall that the indicator function of A is defined by

$$\mathbb{I}_A(x) \;=\; \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in A^c \end{cases}$$

a. Show that $\mathbb{I}_{A^c} = 1 - \mathbb{I}_A$.

b. Show that $\mathbb{I}_A - \mathbb{I}_B = \mathbb{I}_{B^c} - \mathbb{I}_{A^c}$.

c. Show that $\mathbb{I}_{A \cap B} = \mathbb{I}_A \, \mathbb{I}_B$.

d. Let $u, v \in \{0, 1\}$. Show that $\mathbb{I}(u \neq v) = |\mathbb{I}(u = 1) - \mathbb{I}(v = 1)|$. Hint: Consider separately the cases $\mathbb{I}(u \neq v) = 0$ and $\mathbb{I}(u \neq v) = 1$.

3. Let (X, Y) be a jointly distributed pair with $X \in \mathcal{X}$ and $Y \in \{0, 1\}$. Suppose that $\mathcal{X}$ is finite and that (X, Y) has joint probability mass function $p(x, y)$.

a. Express the prior probabilities $\pi_0 = \mathbb{P}(Y = 0)$ and $\pi_1 = \mathbb{P}(Y = 1)$ in terms of $p(x, y)$.

b. Express the class conditional probability mass function $p_0(x) = \mathbb{P}(X = x \,|\, Y = 0)$ in terms of $p(x, y)$ and the prior probabilities.

c. Show that the marginal pmf of X can be written as $p(x) = \pi_0 \, p_0(x) + \pi_1 \, p_1(x)$ where $p_1(x) = \mathbb{P}(X = x \,|\, Y = 1)$.

e. Use Bayes rule to show that $\eta(x) := P(Y = 1 \,|\, X = x) = \pi_1 p_1(x) / p(x)$

4. Let (X, Y) be a discrete random pair with joint probability mass function $p(x, y)$. Recall from the lecture notes that we may define $\mathbb{E}(Y|X) = \varphi(X)$ where $\varphi(x) = \sum_y y\, p(y|x)$. Establish the following.

a. If $Y \geq 0$ then $\mathbb{E}(Y|X) \geq 0$

b. $\mathbb{E}(aY + b|X) = a\,\mathbb{E}(Y|X) + b$

c. $\mathbb{E}\{\,\mathbb{E}(Y|X)\,\} = \mathbb{E}Y$

5. Let X, Y be non-negative random variables with joint density function $f(x, y) = y^{-1}\, e^{-x/y}\, e^{-y}$ for $x, y \geq 0$.

a. Find the marginal density $f(y)$ of Y

b. Find the conditional density $f(x \,|\, y)$ of X given $Y = y$

c. Find $\mathbb{E}[X \,|\, Y = y]$

d. Find $\mathbb{E}[X \,|\, Y]$

6. Let $\mathbf{A}$ and $\mathbf{B}$ be invertible $n \times n$ matrices. Argue that $(\mathbf{A}\,\mathbf{B})^{-1} = \mathbf{B}^{-1}\,\mathbf{A}^{-1}$.

7. Let $\mathbf{A}$ be an $n \times n$ matrix. Show that if $\mathbf{A}$ has rank n then $\mathbf{A}\mathbf{x} = 0$ if and only if $\mathbf{x} = 0$. Hint: If $\mathbf{A}$ has rank n then its columns are linearly independent.

8. Let $A \in \mathbb{R}^{d \times d}$ be symmetric. The spectral theorem tells us that there is an orthonormal basis $v_1, \ldots, v_d$ for $\mathbb{R}^d$ such that each v_i is an eigenvector of A.

a. Show that the $d \times d$ matrix $\Gamma = [v_1, \ldots, v_d]$ is orthogonal, that is $\Gamma^t \Gamma = I$. Note that this implies $\Gamma \Gamma^t = I$, though you do not need to show this.

b. Let $D = \text{diag}(\lambda_1, \ldots, \lambda_d)$ be the $d \times d$ diagonal matrix with D_{ii} equal to the ith eigenvalue of A and all other entries equal to zero. Show that $A\Gamma = \Gamma D$.

c. Conclude from the expression above that A can be written in the form $A = \Gamma D \Gamma^t$