STOR 654 Homework 12

1. Show that if $U \sim \chi^2_n$ with $n \geq 3$ then $\mathbb{E}U^{-1} = 1/(n-2)$.

2. Let $Y \sim \mathcal{N}_n(\theta, \sigma^2 I)$ be generated from a Gaussian sequence model with $\theta \in \mathbb{R}^n$ and σ^2 known. Show that the MLE of θ based on Y is Y itself.

3. Let $\Gamma(x)$ be the standard Gamma function, defined for $x > 0$. Show that if $Z \sim \mathcal{N}(0,1)$ then for each $p \geq 1$

$$\mathbb{E}|Z|^p \;=\; \frac{2^{p/2}}{\sqrt{\pi}}\Gamma((1+p)/2)$$

Deduce from this fact and Stirling's approximation that $||Z||_p := (\mathbb{E}|Z|^p)^{1/p} = O(p^{1/2})$.

4. (Stein's Identity for Covariance) Let $(X,Y)^t \sim \mathcal{N}_2(0,\Sigma)$ be jointly distributed normal random variables, and let f be a continuously differentiable real-valued function satisfying appropriate integrability conditions.

   a. Using the representation theorem, argue that we can write $X = aZ_1 + bZ_2$ and $Y = bZ_1 + cZ_2$ where Z_1, Z_2 are independent standard normal random variables, and a, b, c are real constants.

   b. Find $\text{Cov}(X,Y)$ in terms of a, b, c.

   c. Show that $\text{Cov}(f(X),Y) = \mathbb{E}f'(X)\,\text{Cov}(X,Y)$. Hint: Use the representations of X and Y in terms of Z_1 and Z_2. Apply Stein's identity after appropriate conditioning.

5. (Tensorization) Let $P_1,\ldots,P_n$ and $Q_1,\ldots,Q_n$ be distributions on $\mathcal{X}$. Show that

- $\text{TV}(\otimes_{i=1}^n P_i, \otimes_{i=1}^n Q_i) \;\leq\; \sum_{i=1}^n \text{TV}(P_i, Q_i)$
- $\text{KL}(\otimes_{i=1}^n P_i, \otimes_{i=1}^n Q_i) \;=\; \sum_{i=1}^n \text{KL}(P_i, Q_i)$

6. (Longest increasing subsequence) For $n \geq 1$ let $f_n : [0,1]^n \to \{1,2,\ldots\}$ be the longest increasing subsequence function defined as follows: $f(x_1,\ldots,x_n)$ is the largest integer k for which there exist indices $1 \leq i_1 < \cdots < i_k \leq n$ such that $x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_k}$.

   a. Carefully find the difference coefficients $c_1,\ldots,c_n$ of f_n.

   b. Let $X_1,\ldots,X_n \in [0,1]$ be independent. Find a bound on $\mathbb{P}(f_n(X_1^n) - \mathbb{E}f_n(X_1^n) \geq t)$ when $t \geq 0$.

c. What can you say about the limiting behavior of $\mathbb{E}f_n(X_1^n)$ if $X_1, X_2, \ldots \in [0,1]$ is stationary. Justify your answer.

7. Let $U \sim \mathcal{N}_d(\mu, \Sigma)$ and let $V = \Sigma^{1/2}Y + \mu$ where $Y \sim \mathcal{N}_d(0, I)$.

(a) Show that $\mathbb{E}U = \mathbb{E}V$ and that $\text{Var}(U) = \text{Var}(V)$.

(b) Fix $v \in \mathbb{R}^d$. Find the distributions of the random variables v^tU and v^tV. Note that these distributions are the same.

8. Let X and Y be random variables with distributions $P \sim f$ and $Q \sim g$, respectively. Fix $a > 0$ and let P' and Q' be the distributions of $X' = aX$ and $Y' = aY$.

(a) Does $\text{KS}(P, Q) = \text{KS}(P', Q')$?

(b) Does $\text{TV}(P, Q) = \text{TV}(P', Q')$?