Subadditivity, Bounds on Expected Maxima

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Subadditive Sequences

Defn: A sequence $a_1, a_2, \ldots \in \mathbb{R}$ is *subadditive* if for every $m, n \geq 1$

$$a_{m+n} \leq a_m + a_n$$

Fact: If $\{a_n : n \geq 1\}$ is subadditive, then the limit of a_n/n exists, and

$$\lim_{n \to \infty} \frac{a_n}{n} = \inf_{n \geq 1} \frac{a_n}{n}$$

Definition: A stochastic sequence $X_1, X_2, \ldots \in \mathcal{X}$ is *stationary* if for each $n, j \geq 1$

$$(X_1, \ldots, X_n) \stackrel{\mathrm{d}}{=} (X_{j+1}, \ldots, X_{j+n})$$

In words, the distribution of a block of variables is independent of its location

Example: Uniform Law of Large Numbers

Let $X_1, X_2, \ldots \in \mathcal{X}$ be stationary and let $\mathcal{G}$ be a family of functions $g : \mathcal{X} \to [-c, c]$.
Claim: The sequence

$$a_n \;=\; \mathbb{E} \sup_{g \in \mathcal{G}} \left| \sum_{i=1}^{n} (g(X_i) - \mathbb{E}g(X)) \right|$$

is subadditive. Therefore for some $\alpha \geq 0$,

$$\frac{a_n}{n} \;=\; \mathbb{E} \sup_{g \in \mathcal{G}} \left| \frac{1}{n} \sum_{i=1}^{n} g(X_i) - \mathbb{E}g(X) \right| \;\to\; \alpha$$

Note: If $X_1, X_2, \ldots$ are iid, then we can use this fact in conjunction with McDiarmid's inequality and the Borel-Cantelli lemma to show final supremum converges to α wp1.

Example: Bin Packing

Let $X_1, X_2, \ldots \in [0,1]$ be stationary and let $f_n : [0,1]^n \to \mathbb{N}$ be the bin-packing function for n objects. Claim: The sequence

$$a_n \; = \; \mathbb{E} f_n(X_1, \ldots, X_n)$$

is subadditive. Therefore for some $\alpha \geq 0$

$$\frac{\mathbb{E} f_n(X_1, \ldots, X_n)}{n} \; \to \; \alpha$$

Note: If $X_1, X_2, \ldots$ are iid, then we can use this fact, McDiarmid's inequality, and the Borel-Cantelli lemma to show that $f_n(X_1, \ldots, X_n)/n \to \alpha$ wp1.

Example: Longest Increasing Subsequence

Let $X_1, X_2, \ldots \in [0,1]$ be stationary and let $f_n : [0,1]^n \to \mathbb{N}$ be the longest increasing subsequence function. Claim: The sequence

$$a_n \; = \; \mathbb{E} f_n(X_1, \ldots, X_n)$$

is subadditive. Therefore for some $\alpha \geq 0$

$$\frac{\mathbb{E} f_n(X_1, \ldots, X_n)}{n} \; \to \; \alpha$$

Note: If $X_1, X_2, \ldots$ are iid, then we can use this fact, McDiarmid's inequality, and the Borel-Cantelli lemma to show that $f_n(X_1, \ldots, X_n)/n \to \alpha$ wp1.

MGF Bound on Expected Value of Maximum

Task: Find bounds on $\mathbb{E}\max(X_1,\ldots,X_n)$ *using MGFs of* X_i

Gaussian case: If $X_1,\ldots,X_n \sim \mathcal{N}(0,\sigma^2)$ then

$$\mathbb{E}\max(X_1,\ldots,X_n) \leq \sigma\sqrt{2\log n}$$

General case: If $X_1,\ldots,X_n$ are such that $M_{X_i}(s) \leq M(s)$ for all $s \geq 0$ then

$$\mathbb{E}\max(X_1,\ldots,X_n) \leq \inf_{s:M(s)\geq 1} \frac{\log n + \log M(s)}{s}$$

Note that the random variables X_i need not be independent.