Extreme Value Theory for the Gaussian

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Extreme Value Theory

Setting: Let $X_1, X_2, \ldots \in \mathbb{R}$ be iid with CDF F. Interested in the limiting behavior of $M_n = \max(X_1, \ldots, X_n)$.

Question: Are there scaling and centering constants $\{a_n\}$ and $\{b_n\}$ such that

$$\tilde{M}_n = a_n(M_n - b_n) \text{ has limiting CDF } G? \qquad (\star)$$

Extreme Value Theorem: If ($\star$) holds then $G(x) = G_0(ax + b)$ where a, b are constants and one of the following holds

(1) $G_0(x) = \exp(-e^{-x})$

(2) $G_0(x) = \exp(-x^{-\alpha})\,\mathbb{I}(x > 0)$ some $\alpha > 0$

(3) $G_0(x) = \exp(-(-x)^{\alpha})\,\mathbb{I}(x \leq 0) + \mathbb{I}(x > 0)$ some $\alpha > 0$

Preliminary Result

Fact: Let $X_1, X_2, \ldots \in \mathbb{R}$ be iid with CDF F. Let $M_n = \max(X_1, \ldots, X_n)$ and $\tau \geq 0$. For any sequence $u_1,\ u_2, \ldots \in \mathbb{R}$ the following are equivalent

(1) $n\,(1 - F(u_n)) \to \tau$

(2) $\mathbb{P}(M_n \leq u_n) \to e^{-\tau}$

Maxima of Gaussian Random Variables

Basic question: Given $Z_1, Z_2, \ldots$ iid $\sim \mathcal{N}(0,1)$, interested in the limiting behavior of

$$M_n \ := \ \max(Z_1, \ldots, Z_n)$$

Note: MGF bound shows that $\mathbb{E} M_n \leq \sqrt{2 \log n}$.

Key Tool: Let $\phi(x)$ and $\Phi(x)$ be the pdf and CDF of the standard normal, and let $\overline{\Phi}(x) = 1 - \Phi(x)$. Then for each $x > 0$

$$\frac{\phi(x)}{x} \left(1 - \frac{1}{x^2}\right) \ \leq \ \overline{\Phi}(x) \ \leq \ \frac{\phi(x)}{x}$$

First Results on Gaussian Extremes

Fact: Let $\Phi^{-1}(s)$ be the inverse CDF (percentile function) for $Z \sim \mathcal{N}(0,1)$. Then

$$\frac{\Phi^{-1}(1-t^{-1})}{\sqrt{2\log t}} \to 1 \text{ as } t \to \infty$$

Example: Let $z(\alpha) = \Phi^{-1}(1-\alpha)$ be the upper α percentile of $\mathcal{N}(0,1)$. Fact shows that $z(\alpha)$ grows like $\sqrt{2\log \alpha^{-1}}$ as $\alpha \to 0$.

Fact: If $Z_1, Z_2, \ldots$ be iid $\sim \mathcal{N}(0,1)$ then

$$\frac{\mathbb{E}\max(|Z_1|,\ldots,|Z_n|)}{\sqrt{2\log n}} \to 1 \text{ as } n \to \infty$$

Gaussian Extreme Value Theorem

Define *scaling* constants $\{a_n\}$ and *centering* constants $\{b_n\}$ as follows

$$a_n = \sqrt{2\log n} \qquad b_n = \sqrt{2\log n} - \frac{\log(4\pi \log n)}{\sqrt{8\log n}}$$

Theorem: If $Z_1, Z_2, \ldots$ iid $\sim \mathcal{N}(0,1)$ and $M_n = \max(Z_1, \ldots, Z_n)$ then for $x \in \mathbb{R}$,

$$\mathbb{P}(a_n(M_n - b_n) \leq x) \; \to \; \exp\{-e^{-x}\}$$

Note: Limiting CDF is that of $-\log U$ with $U \sim$ Exp(1)