STOR 654 Homework 9

1. (Calculus Exercise). Let $h(u) = (1+u)\log(1+u) - u$. (This function appears in Bennett's exponential inequality for sums of independent, bounded random variables.)

(a) By considering the first few terms of Taylor expansion of the function $h(\cdot)$ around zero, show that for every $u \geq 0$

$$h(u) \geq \frac{u^2}{2+2u}$$

(b) (Optional) Use calculus to establish the stronger bound that for every $u \geq 0$

$$h(u) \geq \frac{u^2}{2+2u/3}$$

2. Carefully reproduce the arguments in class for Bennett's inequality, including the basic MGF bound, and including the details of the Chernoff bound.

3. Let X be a random variable satisfying the concentration type inequality $\mathbb{P}(|X| > t) \leq a\, e^{-b\, t^2}$ for all $t \geq 0$, where $a \geq 1$ and $b \geq 0$. Show that

$$\mathbb{E}|X| \leq \sqrt{\frac{1+\log a}{b}}.$$

Hint: Begin by showing that for $s \geq 0$, $\mathbb{E}X^2 \leq s + \int_s^\infty \mathbb{P}(X^2 \geq t)\, dt$. Use Cauchy-Schwartz.

4. Let $X_1, \ldots, X_n$ be random variables with moment generating functions $M_{X_i}(s) \leq M(s)$ for each $s \geq 0$.

(a) Using the argument in class for Gaussian random variables, show that

$$\mathbb{E} \max(X_1, \ldots, X_n) \;\leq\; \inf_{s>0} \frac{\log n + \log \varphi(s)}{s}.$$

Suppose now that $U_1, \ldots, U_n$ are Gamma(α, β) random variables. Note that the moment generating function of U_i is $\varphi(s) = (1 - s\,\beta)^{-\alpha}$.

(b) Using the bound from part (a) and an appropriate choice of s, which can be found by inspection, show that

$$\mathbb{E} \max(U_1, \ldots, U_n) \;\leq\; \frac{2\,\beta\, \log n}{1 - n^{-1/\alpha}}.$$

5. Show that if $X \sim \mathcal{N}_d(\mu, \Sigma)$ and $U = X^T A X$ then $\mathbb{E}U = \text{tr}(A\Sigma) + \mu^T A \mu$. (It may be helpful to use the fact that $\text{tr}(UV) = \text{tr}(VU)$.)

6. *Concentration for norms of Gaussian random vectors.* Let $Y \sim \mathcal{N}_d(0, \Sigma)$ and consider the random variable $U = ||Y||$.

(a) Show that $U = F(X)$ in distribution, where $X \sim \mathcal{N}_d(0, I)$ and $F(x) = ||\Sigma^{1/2} x||$

(b) Show that F is Lipschitz with constant

$$L \;\leq\; \sup_{u \in \mathbb{R}^d \setminus \{0\}} \frac{||\Sigma^{1/2} u||}{||u||}$$

(c) Express the right hand side of the inequality above in terms of the eigenvalues of Σ.

(d) Find a concentration inequality for U.

7. Let U and V be independent $\mathcal{N}(0, 1)$ random variables. Define $Y = V$ and let

$$X \;=\; \begin{cases} U & \text{if } UV \geq 0 \\ -U & \text{if } UV < 0 \end{cases}$$

(a) Show that X and Y each have a standard normal distribution, but that (X, Y) is not bivariate normal.

(b) Show that X^2 and Y^2 are independent.

8. Let $\Phi : \mathbb{R} \to (0, 1)$ be the CDF of the standard normal, and let $\Phi^{-1} : (0, 1) \to \mathbb{R}$ be it's inverse function, equivalently, the percentile function of the standard normal.

(a) Find the limit of $\Phi^{-1}(\alpha)$ as $\alpha \to 0$ and $\alpha \to 1$.

To simplify notation in what follows, let $s(t) = \sqrt{2 \log t}$ for $t \geq 1$.

(b) Use bounds on $\overline{\Phi}(s)$ to show that

$$\limsup_{t \to \infty} t\, \overline{\Phi}(s(t)) \leq 1.$$

(c) Use bounds on $\overline{\Phi}(s)$ to show that for every $\delta \in (0, 1)$

$$\liminf_{t \to \infty} t\, \overline{\Phi}(\delta\, s(t)) \geq 1.$$

(d) Combine the bounds from (b) and (c) to show that as

$$\liminf_{t\to\infty} \frac{\Phi^{-1}(1-t^{-1})}{\sqrt{2\,\log t}} = 1$$

9. Let $X_1, \ldots, X_n$ be independent standard normal random variables. Here we identify upper and lower bounds for the expectation of $K_n := \max_{1\le i\le n} |X_i|$.

(a) Using the MGF-based bound from class and the fact that $K_n = \max_i(X_i, -X_i)$ show that $\mathbb{E}K_n \le (2\,\log 2n)^{1/2}$.

(b) Let Φ^{-1} be the inverse CDF (percentile function) of the standard normal. Show that

$$K_n \;=\; \Phi^{-1}\left(\frac{1}{2} + \frac{1}{2}\max_{1\le i\le n} V_i\right)$$

where $V_1, \ldots, V_n$ are independent Uniform$(0,1)$ random variables.

(c) Show that $\Phi^{-1}(u)$ is convex on $[1/2, 1)$. Apply Jensen's inequality to the expression in (b) to obtain the bound $\mathbb{E}K_n \ge \Phi^{-1}\left(1 - 1/(2n+2)\right)$.

(d) Conclude from (a), (c), and the previous problem that $\mathbb{E}K_n/\sqrt{2\,\log n} \to 1$ as $n \to \infty$.

10. Let $X \sim \mathcal{N}_p(\mu, \Sigma)$. Show that $Y = AX$ and $Z = BX$ are independent iff $A\Sigma B^t = 0$. Remember to check that Y and X are jointly normal.