STOR 654 Homework 8

1. Establish the following relations for random vectors X and Y, matrices A, and vectors b of appropriate dimension.

(a) $\mathbb{E}(AX + b) = A\,\mathbb{E}X + b$

(b) $\text{Var}(X + b) = \text{Var}(X)$

(c) $\text{Var}(AX) = A\,\text{Var}(X)A^t$

(d) $\text{Cov}(X, Y) = \text{Cov}(Y, X)^t$

(e) $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + \text{Cov}(X, Y) + \text{Cov}(Y, X)$

(f) If X, Y are independent, then $\text{Cov}(X, Y) = 0$

(g) $\text{Var}(X)$ is non-negative definite

2. Let U_1 and U_2 be independent random variables with mean zero and variance one, and let $U_3 = U_1 + 3U_2$ and $U_4 = 2U_1 - U_2$. Define random vectors $X = (U_1, \ldots, U_4)^t$, $Y = (U_1, U_2)^t$, and $Z = (U_3, U_4)^t$. Note that $X = (Y^t, Z^t)^t$. Find the following

(a) $\text{Var}(X)$

(b) $\text{Var}(Y)$

(c) $\text{Var}(Z)$

(d) $\text{Cov}(Y, Z)$

(e) $\text{Cov}(Z, Y)$

Note that the matrices in (b) - (e) correspond to block submatrices of the variance matrix you found in (a). Which matrices correspond to diagonal blocks, and which matrices correspond to off-diagonal blocks? Discuss.

3. (Bin packing) For $n \geq 1$ let $f_n : [0, 1]^n \to \{0, 1, 2, \ldots\}$ be the bin packing function for n objects, that is, $f_n(x_1, \ldots, x_n)$ is the minimum number of length-1 bins needed to hold objects of length $x_1, \ldots, x_n$.

a. Carefully find the difference coefficients $c_1, \ldots, c_n$ of f_n.

b. Let $X_1, \ldots, X_n \in [0,1]$ be independent. Find a bound on $\mathbb{P}(f_n(X_1^n) - \mathbb{E}f_n(X_1^n) \geq t)$ when $t \geq 0$.

c. Now let $x_1, x_2, \ldots \in [0,1]$ and define $a_n = f_n(x_1^n)$. Is the sequence $\{a_n : n \geq 1\}$ subadditive? Justify your answer.

d. What can you say about the limiting behavior of $\mathbb{E}f_n(X_1^n)$ if $X_1, X_2, \ldots \in [0,1]$ is stationary. Justify your answer.

4. Let $X_1, \ldots, X_n$ be independent Bernoulli random variables with $\mathbb{E}X_i = p_i$. Let $S = X_1 + \cdots + X_n$ and let $\mu = \mathbb{E}S = \sum_{i=1}^n p_i$. Use Chernoff's bound and a MGF computation to show that for all $t > \mu$

$$\mathbb{P}(S > t) \leq \exp\{t - \mu - t\log(t/\mu)\}$$

How does this bound compare to Hoeffding's inequality?

5. Let $\Phi(x)$ and $\phi(x)$ be the cumulative distribution function and density, respectively, of the standard normal distribution. In this problem, you are asked to find a useful approximation to $1 - \Phi(x)$ when x is large. Note that for $x > 0$,

$$1 - \Phi(x) \;=\; \Phi(-x) \;=\; \int_{-\infty}^{-x} \frac{1}{t} \cdot t\,\phi(t)\,dt$$

(a) Apply integration-by-parts to the last integral above. Use the resulting expression establish the upper bound $1 - \Phi(x) \leq x^{-1}\,\phi(x)$ for $x > 0$.

(b) Apply the same steps to the integral appearing in the integration-by-parts. Use this to establish the lower bound

$$1 - \Phi(x) \geq (\frac{1}{x} - \frac{1}{x^3})\,\phi(x) \quad \text{for} \quad x > 0.$$

(c) Conclude that as $x \to \infty$, $(1 - \Phi(x)) = \frac{\phi(x)}{x}(1 + o(1))$

6. (Hoeffding's MGF Bound) Let X be a discrete random variable with pmf $p(\cdot)$. Assume that $a \leq X \leq b$ for a, b finite, and that $\mathbb{E}X = 0$. Let $M_X(s) = \mathbb{E}e^{sX}$ be the moment generating function of X and define $\varphi(s) := \log M_X(s)$.

a. Show that

$$\varphi'(s) = \frac{\mathbb{E}[Xe^{sX}]}{\mathbb{E}e^{sX}} \quad \text{and} \quad \varphi''(s) = \frac{\mathbb{E}[X^2e^{sX}]}{\mathbb{E}e^{sX}} - (\varphi'(s))^2$$

b. Verify that $\varphi(0) = \varphi'(s) = 0$

Now fix $t > 0$ and let U be a new random variable having the "exponentially tilted" pmf

$$q(x) \;=\; \frac{p(x)e^{tx}}{\mathbb{E}e^{tX}}$$

c. Verify that $q(\cdot)$ is a pmf and that $a \le U \le b$

d. Show that $\mathbb{E}(U) = \varphi'(t)$ and that $\mathrm{Var}(U) = \varphi''(t)$.

e. Using the variance bound for bounded random variables, conclude from (c) and (d) that $\varphi''(t) \le (b-a)^2/4$.

f. Argue that for $s > 0$, $\varphi(s) \le s^2(b-a)^2/8$. Exponentiating gives Hoeffding's MGF bound.