STOR 654 Homework 7

1. Let X, Y be random variables and $a, b > 0$. Show that

   (a) $\mathbb{P}(|X+Y| \geq a+b) \leq \mathbb{P}(|X| \geq a) + \mathbb{P}(|Y| \geq b)$

   (b) $\mathbb{P}(|XY| \geq a) \leq \mathbb{P}(|X| \geq a/b) + \mathbb{P}(|Y| \geq b)$

2. Find the moment generating function of $X \sim \mathcal{N}(0, \sigma^2)$ and $X \sim \text{Poiss}(\lambda)$.

3. Show that if $X_1, \ldots, X_n$ are independent and each MGF $M_{X_i}(s)$ is well defined in a neighborhood of 0 then $S = X_1 + \cdots + X_n$ has MGF

$$M_S(s) \;=\; \prod_{i=1}^{n} M_{X_i}(s).$$

4. Recall that the chi-squared distribution with k degrees of freedom, denoted χ^2_k, has a $\Gamma(k/2, 2)$ density.

   (a) Show that if $Z \sim \mathcal{N}(0,1)$ then $Z^2 \sim \chi^2_1$.

   (b) Show that if $Z_1, \ldots, Z_k \sim \mathcal{N}(0,1)$ are iid then $Z_1^2 + \cdots + Z_k^2 \sim \chi^2_k$ (you may appeal to basic properties of the gamma distribution). Use this representation to find the mean and variance of the χ^2_k distribution.

   (c) Use the identity (from a previous HW)

$$\mathbb{E}\exp\{aX^2 + bX\} \;=\; \frac{1}{\sqrt{1-2a\sigma^2}} \exp\left\{\frac{\sigma^2 b^2}{2(1-2a\sigma^2)}\right\}$$

   for $X \sim \mathcal{N}(0, \sigma^2)$ to find the moment generating function of the χ^2_k distribution.

5. Let X be a non-negative random variable such that $\mathbb{E}X^2$ is finite. Show that for each $0 < \lambda < 1$ we have the inequality

$$\mathbb{P}(X \geq \lambda\,\mathbb{E}X) \;\geq\; (1-\lambda)^2\,\frac{(\mathbb{E}X)^2}{\mathbb{E}X^2}$$

Hint: Use the Cauchy-Schwartz inequality and the identity $X = X\,\mathbb{I}(X \geq c) + X\,\mathbb{I}(X < c)$.

6. Let $\mathcal{G}$ be a finite family of functions $g : \mathcal{X} \to [-c, c]$ and let $X_1, \ldots, X_n \in \mathcal{X}$ be iid. Use the union bound and Hoeffding's inequality to find an upper bound on

$$\mathbb{P}\left(\max_{g \in \mathcal{G}} \left| \frac{1}{n} \sum_{i=1}^{n} g(X_i) - \mathbb{E}g(X_1) \right| \geq t \right)$$

7. Let $X_1, \ldots, X_n$ be iid $\sim$ Bern(p). Note that $|X_i - p| \leq \max(p, 1-p)$.

(a) Use Bernstein's inequality to get an upper bound on $\mathbb{P}(n^{-1} \sum_{i=1}^{n} X_i - p \geq t)$ for $t \geq 0$.

(b) Argue that one can restrict attention to $t \in [0, 1-p]$. Using this fact and the bound in part (a) show that if $p \geq 1/2$ then for all $t \geq 0$

$$\mathbb{P}\left(\frac{1}{n} \sum_{i=1}^{n} X_i - p \geq t \right) \; \leq \; \exp\left\{ \frac{-3nt^2}{8p(1-p)} \right\}$$

(c) Compare the bound in part (b) to a naive inequality based on the central limit theorem and tail bounds for the standard normal distribution.

8. Let $X_1, \ldots, X_n \in \mathbb{R}^d$ be independent random vectors such that $\mathbb{E}X_i = 0$ and $||X_i|| \leq c_i/2$ with probability one, where $||u|| = (u^t u)^{1/2}$ is the ordinary Euclidean norm. Let $\alpha = (1/4) \sum_{i=1}^{n} c_i^2$.

(a) Show that $\mathbb{E}|| \sum_{i=1}^{n} X_i || \leq \sqrt{\alpha}$.

(b) Use the bounded difference inequality and the inequality in part (a) to show that for all $t \geq \sqrt{\alpha}$

$$P\left(|| \sum_{i=1}^{n} X_i || > t \right) \; \leq \; \exp\left\{ -\frac{(t - \sqrt{\alpha})^2}{2\alpha} \right\}$$

9. Let X, Y be a random variables with finite second moment. Use the Cauchy-Schwartz inequality to show that $\mathrm{SD}(X+Y) \leq \mathrm{SD}(X) + \mathrm{SD}(Y)$.

10. Let $f : \mathcal{X}^n \to \mathbb{R}$ be a function with difference coefficients $c_1, \ldots, c_n$. Show that

$$\sup_{x, y \in \mathcal{X}^n} |f(x) - f(y)| \; \leq \; \sum_{i=1}^{n} c_i$$

11. Let X and Y be independent random variables with $Y > 0$. Find equalities or inequalities relating the following quantities (you may assume all expecations are finite).

(a) $\mathbb{E}X^3$ and $\mathbb{E}X\,\mathbb{E}X^2$

(b) $\mathbb{E}(X/Y)$ and $\mathbb{E}X/\mathbb{E}Y$

(c) $\mathbb{E}(Y\log Y)$ and $\mathbb{E}Y\log\mathbb{E}Y$

(d) $\mathbb{E}(Y\log Y)$ and $\mathbb{E}Y(\mathbb{E}\log Y)$

12. Let $X_1, \ldots, X_n$ be independent with $\mathbb{E}X = 0$ and $|X_i| \leq c$. Show that if $t \geq n^{-1}\sum_{i=1}^n \mathrm{Var}(X_i)$, then

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n X_i \geq t\right) \leq \exp\left\{\frac{-nt}{2+2c/3}\right\}$$