STOR 654 Homework 5

1. Consider a family $\mathcal{P} = \{f(x|\theta) : \theta \in \Theta\}$ of densities with $\Theta = \mathbb{R}$. Given $X \sim f(x|\theta) \in \mathcal{P}$ we wish to find a Bayesian point estimate of θ based on prior density $\pi(\theta)$.

(a) Show that under absolute loss $\ell(\theta, \theta') = |\theta - \theta'|$, the Bayes estimator is the posterior median

$$\hat{\theta}_\pi(x) \;=\; u \text{ such that } \int_{-\infty}^{u} \pi(\theta|x)\, d\theta = \frac{1}{2}$$

You may assume the median exists.

(b) Show that under zero-one loss $\ell(\theta, \theta') = \mathbb{I}(\theta \neq \theta')$, the Bayes estimator is the posterior mode

$$\hat{\theta}_3(x) \;=\; \operatorname*{argmax}_{\theta \in \Theta} \pi(\theta|x)$$

You may assume the maximum exists.

2. Let $X \sim \text{Bin}(n, \theta)$ with $\theta \in (0, 1)$. Suppose that we wish to estimate θ under the loss function

$$\ell(\theta, \theta') = \frac{(\theta - \theta')^2}{\theta(1 - \theta)}$$

Consider the natural estimator $\hat{\theta}(x) = x/n$.

(a) Find the risk of $\hat{\theta}(x)$ under the loss ℓ.

Suppose that we place the (uniform) prior $\pi(\theta) = 1$ on the unknown parameter θ.

(b) Find the marginal distribution $p(x)$ of X and the the posterior distribution $\pi(\theta|x)$ of θ given $X = x$ under π.

(c) By considering posterior risk, show that $\hat{\theta}$ is the Bayes estimator under π.

(d) What can you say about the minimaxity of $\hat{\theta}$?

Hint: In parts (b) and (c) you will want to make use of the Beta distribution.

3. Let $\mathcal{P} = \{f(x|\theta) : \theta \in \Theta\}$ be a family of distributions on a sample space $\mathcal{X}$, and let $T : \mathcal{X} \to \mathcal{T}$ be a sufficient statistic for $\mathcal{P}$. Show that if $T = \phi(S)$ is a function of some statistic $S : \mathcal{X} \to \mathcal{S}$ then S is also sufficient for $\mathcal{P}$.

4. (Uniqueness of MVUEs)

(a) Establish that for each $a, b \in \mathbb{R}$ we have $(a/2 + b/2)^2 \leq a^2/2 + b^2/2$ with equality iff $a = b$. (This follows from the strict convexity of the function $f(x) = x^2$, but give a direct argument.)

Let $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ and let $\hat{\theta}_1$ and $\hat{\theta}_2$ be minimum variance unbiased estimators of some function $\tau(\theta)$. Define

$$\tilde{\theta} = \frac{1}{2}\hat{\theta}_1 + \frac{1}{2}\hat{\theta}_2 \quad \text{and} \quad W = \frac{1}{2}\hat{\theta}_1(X)^2 + \frac{1}{2}\hat{\theta}_2(X)^2 - \tilde{\theta}(X)^2$$

(b) Show that $W \geq 0$ and that $\mathbb{E}_\theta W \leq 0$ for each θ. Conclude that $\hat{\theta}_1 = \hat{\theta}_2$ with P_θ-probability one for each θ.