STOR 654 Homework 3

1. Show that the Poisson and Gamma families of distributions are canonical exponential families. Make sure to identify the base function h, the sufficient statistic T, the log-partition function A, the natural parameter space $\mathcal{H}$, and the relationship between the standard and canonical parameters.

2. Let $\mathcal{P}$ be a canonical E-F generated by h, T with natural parameter space $\mathcal{H}$ and log-partition function A. Show that for all $1 \leq j, l \leq k$, $\frac{\partial^2}{\partial \eta_j \partial \eta_l} A(\eta) = \mathrm{Cov}_\eta(T_j(X), T_l(X))$.

3. Let X be a random variable with finite second moment. Give two proofs of the elementary inequality $\mathbb{E}|X| \leq \sqrt{\mathbb{E}X^2}$: one using Jensen's inequality; and one using Cauchy-Schwartz.

4. Show that if $X \in \mathbb{R}$ has a continuous, strictly increasing CDF F then $F(X) \sim \mathrm{U}(0,1)$ is uniform.

5. Let $X \sim \mathcal{N}(0, \sigma^2)$. Establish the identity

$$\mathbb{E} \exp\{aX^2 + bX\} \;=\; \frac{1}{\sqrt{1 - 2a\sigma^2}} \exp\left\{ \frac{\sigma^2 b^2}{2(1 - 2a\sigma^2)} \right\}$$

Hint: Write the expectation as an integral. Combine terms in the exponent and complete the square. Remove the constant factor and perform a simple change of variables to evaluate the remaining integral.

6. Find the density of $Y = -\log X$ where $X \sim \mathrm{Exp}(1)$.

7. (Incomplete beta function) Let $\mathrm{Bin}(n,p)$ denote the binomial distribution with parameters $n \geq 1$ and $p \in [0,1]$. Show that for each $1 \leq k \leq n$ and each $p \in [0,1]$ that the following identity holds:

$$P(\mathrm{Bin}(n,p) \geq k) \;=\; \frac{n!}{(k-1)!(n-k)!} \int_0^p u^{k-1}(1-u)^{n-k} du$$

Hint: Fix $1 \leq k \leq n$. Let $f(p)$ and $g(p)$ be, respectively, the left- and right-hand sides of the equation. Show that f, g are equal when $p = 0$. Then show that $f'(p) = g'(p)$ for each $p \in (0,1]$.

8. Let X be a random variable taking values in the finite interval $[0, c]$.

(a) Show that $EX \leq c$ and $EX^2 \leq cEX$.

(b) Use these inequalities to show that

$$\text{Var}(X) \leq c^2[u(1-u)] \quad \text{where} \quad u = \frac{EX}{c} \in [0, 1].$$

(c) Use the result of part (b) to show that $\text{Var}(X) \leq c^2/4$.

(d) Show that this bound is achieved, that is, find a random variable $X \in [0, c]$ for which $\text{Var}(X) = c^2/4$. Hint: put the probability mass of X at the endpoints of the interval.

(e) Use the result in (c) to bound the variance of a random variable X taking values in an interval $[a, b]$ with $-\infty < a < b < \infty$.

9. Let $U_1, \ldots, U_n$ be independent Uniform$(0, \theta)$ random variables. Find $\mathbb{E}\left[\max_{1 \leq j \leq n} U_j\right]$.

10. Let f be a convex function on an open interval $I \subseteq \mathbb{R}$ and let $b \in I$.

(a) Show that for $a, c \in I$ such that $a < b < c$ we have

$$L(a, b) := \frac{f(b) - f(a)}{b - a} \leq \frac{f(c) - f(a)}{c - a} \leq \frac{f(c) - f(b)}{c - b} := U(c, b). \tag{1}$$

(Hint: express b as a convex combination of a and c and then apply the definition of convexity.)

(b) Let $L^*(b) = \sup_{a<b} L(a, b)$ and $U^*(b) = \inf_{c>b} U(c, b)$. Using equation (1) above, argue carefully that $L^*(b) \leq U^*(b)$ and that both quantities are finite.

(c) Argue that for every $c \in I$ with $c > b$ the inequality $f(c) \geq f(b) + (c - b)\, L^*(b)$ holds. Argue that for every $a \in I$ with $a < b$ the inequality $f(a) \geq f(b) + (a - b)\, U^*(b)$ holds.

(d) Let u be any number in the interval $[L^*(b), U^*(b)]$, which is non-empty by part (b). Show that u is a subgradient for f at b.

11. (Variational characterization of the median) Let X be a random variable with density f and finite expectation, and let M be a median of X. We wish to establish that

$$M \;=\; \operatorname*{argmin}_{a\in\mathbb{R}} \mathbb{E}|X-a|$$

or equivalently that

$$\mathbb{E}|X-M| \;\leq\; \mathbb{E}|X-a| \text{ for all } a\in\mathbb{R}.$$

a. Replacing X by $X-M$, we may assume without loss of generality that $M=0$. Let $a>0$. Express the difference $\mathbb{E}|X-a|-\mathbb{E}|X|$ as a sum of integrals over the disjoint intervals $(-\infty,0]$, $(0,a]$, and (a,∞). By carefully considering each integral, show that

$$\mathbb{E}|X-a|-\mathbb{E}|X| \;\geq\; a\,\{\,\mathbb{P}(X\leq 0)-\mathbb{P}(0<X\leq a)-\mathbb{P}(X>a)\,\}\,.$$

Use the definition of the median and the fact that $a\geq 0$ to conclude that the right side of the inequality above is non-negative. [A similar argument can be carried out for $a\leq 0$, but you do not need to do this.]

b. Suppose now that X has finite variance. Using the variational characterization of the median with $a=\mathbb{E}X$ and Jensen's inequality, show that $|\mathbb{E}X-M|\leq\sqrt{\mathrm{Var}(X)}$.

12. (Saddle points and minimax) Let $\mathcal{X}$ and $\mathcal{Y}$ be sets and let $f:\mathcal{X}\times\mathcal{Y}\to\mathbb{R}$ be any function.

a. Show that, with no further assumptions,

$$\sup_{y\in\mathcal{Y}}\inf_{x\in\mathcal{X}} f(x,y) \;\leq\; \inf_{x\in\mathcal{X}}\sup_{y\in\mathcal{Y}} f(x,y) \tag{2}$$

This simple fact plays an important role in optimization, where it implies the weak duality property of the Lagrange dual problem, and in game theory, where it has connections with Nash equilibria. A pair $(\tilde{x},\tilde{y})\in\mathcal{X}\times\mathcal{Y}$ is called a *saddle point* for f if

$$f(\tilde{x},y) \;\leq\; f(\tilde{x},\tilde{y}) \;\leq\; f(x,\tilde{y}) \quad \text{for every } x\in\mathcal{X} \text{ and } y\in\mathcal{Y}$$

b. Show that if $(\tilde{x},\tilde{y})$ is a saddle point for f then

$$f(\tilde{x},\tilde{y}) \;=\; \inf_{x\in\mathcal{X}} f(x,\tilde{y}) \quad \text{and} \quad f(\tilde{x},\tilde{y}) \;=\; \sup_{y\in\mathcal{Y}} f(\tilde{x},y)$$

How do these inequalities explain the use of the terminology "saddle point"? Hint: assume that f is nice and smooth, and sketch what it will look like in a neighborhood around the point $(\tilde{x},\tilde{y})$.

c. Show that the existence of a saddle point implies equality in inequality (2) above.