STOR 654 Homework 2

1. Use Hölder's inequality to show that $\log \Gamma(u)$ is convex, where $\Gamma(u)$ is the usual Gamma function.

2. The *gamma distribution* with parameters $\alpha, \beta > 0$, denoted by $\Gamma(\alpha, \beta)$, has density

$$g_{\alpha,\beta}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad x \geq 0$$

   (a) Argue that $g_{\alpha,\beta}$ is in fact a density, i.e., that it is non-negative and integrates to 1.

   (b) Show that if $0 < \alpha < 1$ then $g_{\alpha,\beta}$ is convex and that $g_{\alpha,\beta}(x) \to \infty$ as $x \to 0$.

   (c) Show that if $\alpha > 1$ then $g_{\alpha,\beta}$ is unimodal with maximum at $x = (\alpha - 1)/\beta$.

   (d) Let $X \sim \Gamma(\alpha, \beta)$. Show that $\mathbb{E}X = \alpha/\beta$, $\text{Var}(X) = \alpha/\beta^2$.

   (e) Show that if $X \sim \Gamma(\alpha, \beta)$ then $sX \sim \Gamma(\alpha, \beta/s)$ when $s > 0$. Check that this is consistent with the formulas for the expectation and variance of X above.

3. Let $Z_1, \ldots, Z_n$ be iid $\mathcal{N}(0, 1)$.

   a. Find the density of Z_1^2. Argue that the density is in the Gamma family.

   b. It is a basic fact that if $X \sim \text{Gamma}(\alpha_1, \beta)$ and $Y \sim \text{Gamma}(\alpha_2, \beta)$ are independent then $X + Y \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$. Use this to find the density of $Z_1^2 + \cdots Z_n^2$.

   c. (Optional) Establish the fact above.

4. Let $X_1, \ldots, X_n \in \mathbb{R}$ be iid samples having CDF F and density f.

   a. Show that the rth order statistic $X_{(r)}$ has CDF

$$F_{(r)}(u) \;=\; \sum_{t=r}^{n} \binom{n}{r} F(u)^r (1 - F(u))^{n-r}$$

   b. Show that the rth order statistic $X_{(r)}$ has density

$$f_{(r)}(u) = r \binom{n}{r} F(u)^{r-1} (1 - F(u))^{n-r} f(u)$$

5. (Stein's Lemma) Let $Z \sim \mathcal{N}(0,1)$, and let $f : \mathbb{R} \to \mathbb{R}$ have derivative f'. Stein's Lemma states that if $\mathbb{E}|f'(Z)|$ is finite then $\mathbb{E}(Zf(Z)) = \mathbb{E}f'(Z)$.

a. Establish Stein's Lemma using integration by parts in the special case that f is zero outside a finite interval.

b. Use the basic version of Stein's Lemma to show that if $X \sim \mathcal{N}(\mu, \sigma^2)$ and $\mathbb{E}|f'(Z)|$ is finite then $\mathbb{E}((X-\mu)f(X)) = \sigma^2 \mathbb{E}f'(X)$.

c. Use Stein's Lemma to show (by induction) that if $X \sim \mathcal{N}(0, \sigma^2)$ then $\mathbb{E}X^k = 0$ when k odd, and for all $k \geq 1$

$$\mathbb{E}X^{2k} = \sigma^{2k} \prod_{l=1}^{k} (2l-1)$$

6. The *convolution* of two densities f and g on $\mathbb{R}$, denoted $f * g$, is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy.$$

(a) Show that $(f * g)(x)$ is a density, namely it is non-negative and integrates to one. Make use of Fubini's theorem, allowing the change of order of integration, to establish the latter fact.

(b) Show that if $X \sim f$ and $Y \sim g$ are independent random variables, then

$$P(X + Y \leq v) = \int_{-\infty}^{v} (f_X * f_Y)(u)\, du$$

for every $v \in \mathbb{R}$. Use Fubini's theorem and indicator functions as needed.

7. (The quantile function and defining the median) Let X be a real-valued random variable with CDF $F(x)$. For $0 < p < 1$ define the quantile function

$$\varphi(p) = \inf\{x : F(x) \geq p\}$$

a. Show that for fixed $p \in (0,1)$ the set $\{x : F(x) \geq p\}$ is equal to one of the intervals $(\varphi(p), \infty)$ or $[\varphi(p), \infty)$.

b. Use the right-continuity of F to show that, in fact, $\{x : F(x) \geq p\} = [\varphi(p), \infty)$. Conclude from this that $\varphi(p) \leq x$ if and only if $p \leq F(x)$.

A number M is said to be a *median* of X if $P(X > M) \leq 1/2$ and $P(X < M) \leq 1/2$. Note that X may have more than one median.

c. Argue that $M = \varphi(1/2)$ is a median of X. One way to do this is show that if either condition defining a median is violated then one arrives at a contradiction.

d. Show that $M(X)$ is unique if F is monotone increasing.

8. Show that for $a, b > 0$ we have $\sqrt{a/2} + \sqrt{b/2} \leq \sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$.

9. (Variational characterization of the mean and variance) Let X be a random variable such that $\mathbb{E}X^2$ is finite.

a. Establish the identity $\mathbb{E}(X-a)^2 = \mathbb{E}(X - \mathbb{E}X)^2 + (\mathbb{E}X - a)^2$

b. Conclude that

$$\operatorname*{argmin}_{a \in \mathbb{R}} \mathbb{E}(X-a)^2 \; = \; \{\, \mathbb{E}X \,\} \quad \text{and} \quad \min_{a \in \mathbb{R}} \mathbb{E}(X-a)^2 \; = \; \operatorname{Var}(X)$$

10. Let U, V be random variables with joint distribution function $F(u,v) := \mathbb{P}(U \leq u, V \leq v)$, and individual distribution functions F_U and F_V, respectively.

a. Show that $\lim_{v \to \infty} F(u,v) = F_U(u)$. You may use the following fact: if $A_1 \subseteq A_2 \subseteq \cdots$ are nested events in a probability space then $\mathbb{P}(A_n) \to \mathbb{P}(\cup_{j \geq 1} A_j)$ as n tends to infinity.

b. Show that if the joint distribution function F is continuous then individual distribution functions F_U and F_V are continuous as well.

11. Show that $\log x \leq 2(\sqrt{x} - 1)$ for $x \geq 0$