STOR 654 Homework 1

1. Recall that for real numbers a and b, we say $a \leq b$ if $b - a \geq 0$. Using this definition, carefully establish each of the following elementary facts.

(a) If $a \leq b$ then $-b \leq -a$.

(b) If $a \leq b$ and $c \leq d$ then $a + c \leq b + d$.

(c) If $0 \leq a \leq b$ and $0 \leq c \leq d$ then $ac \leq bd$.

2. Let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ be two sequences of numbers. Establish the following.

(a) $-\min\{a_i\} = \max\{-a_i\}$.

(b) $\max\{a_i + b_i\} \leq \max\{a_i\} + \max\{b_i\}$

(c) $\max\{a_i\} - \max\{b_i\} \ \leq \ \max\{a_i - b_i\}$

3. Let $\{a_\lambda : \lambda \in \Lambda\}$ and $\{b_\lambda : \lambda \in \Lambda\}$ be two, possibly infinite, collections of numbers. Establish the following.

(a) $-\inf_{\lambda \in \Lambda} a_\lambda = \sup_{\lambda \in \Lambda}(-a_\lambda)$.

(b) $\sup_{\lambda \in \Lambda}\{a_\lambda + b_\lambda\} \leq \sup_{\lambda \in \Lambda} a_\lambda + \sup_{\lambda \in \Lambda} b_\lambda$

Does the sup analog of $\max\{a_i\} - \max\{b_i\} \ \leq \ \max\{a_i - b_i\}$ hold?

4. For $x = (x_1, \ldots, x_d)^t \in \mathbb{R}^d$ let $||x|| = (x_1^2 + \cdots x_d^2)^{1/2}$ be the usual Euclidean norm.

(a) Show that $||x|| \geq 0$ with equality if and only if $x = 0$.

(b) Show that for $a \in \mathbb{R}$, $||a\,x|| = |a|\,||x||$, where $ax = (ax_1, \ldots, ax_d)$.

(c) Show that $|\,||x|| - ||y||\,| \ \leq \ ||x - y||$. Hint: Use the triangle inequality to show that $||x|| \leq ||x - y|| + ||y||$, then reverse the roles of x and y.

(d) Show that for $1 \leq i \leq d$,

$$|x_i| \ \leq \ ||x|| \ \leq \ |x_1| + \cdots + |x_d|.$$

Hint: Use the fact that if $a, b \geq 0$ then $a \leq b$ if and only if $a^2 \leq b^2$.

5. Show that the following subsets of $\mathbb{R}^d$ are convex.

(a) $B(x_0, r) := \{x : ||x - x_0|| < r\}$, the open ball of radius $r > 0$ centered at $x_0 \in \mathbb{R}^d$

(b) $\partial H(w, b) = \{x : w^t x = b\}$ where $w \in \mathbb{R}^d$ is a direction and $b \in \mathbb{R}$ is an offset

(c) $\{x : Ax \geq b\}$ where $A \in \mathbb{R}^{k \times d}$, $b \in \mathbb{R}^k$, and $\geq$ is understood componentwise.

6. Show that if C_λ, $\lambda \in \Lambda$ are convex subsets of $\mathbb{R}^d$ then so is their intersection $\cap_{\lambda \in \Lambda} C_\lambda$.

7. Convex and concave functions

(a) Show that if f_λ, $\lambda \in \Lambda$ are convex functions defined on the same convex set C then $f = \sup_{\lambda \in \Lambda} f_\lambda$ is convex.

(b) Use the triangle inequality to show that $f(x) = |x|$ is convex.

(c) Use the second derivative condition to establish the convexity or concavity of $x \log x$ and $\log x$ for $x > 0$. Sketch both functions.

(d) Let A be a bounded subset of $\mathbb{R}^d$. Show that the function $f(x) = \inf_{u \in A} \langle x, u \rangle$ is concave.

8. Establish the following properties of the Gamma function $\Gamma(\alpha) = \int_0^\infty x^{\alpha - 1} e^{-x}$, defined for $\alpha > 0$.

(1) $\Gamma(1) = 1$

(2) Use integration by parts to show that $\Gamma(\alpha + 1) = \alpha \, \Gamma(\alpha)$

(3) $\Gamma(n) = (n - 1)!$

9. Recall that if $f : \mathcal{X} \to \mathbb{R}$ is a real-valued function then the argmax of f is the set of points in x at which f is maximized,

$$\operatorname*{argmax}_{x \in \mathcal{X}} f(x) = \left\{ x \in \mathcal{X} : f(x) = \sup_{u \in \mathcal{X}} f(u) \right\}.$$

Note that the argmax of f is empty if its maximum value is not achieved. The argmin of f is similarly defined.

(a) Let $f : \mathcal{X} \to \mathbb{R}$ be defined on a set $\mathcal{X} \subseteq \mathbb{R}$ by $f(x) = x^2$. Identify the value of

$$\sup_{x \in \mathcal{X}} f(x) \quad \text{and} \quad \operatorname{argmax}_{x \in \mathcal{X}} f(x)$$

in each of the following cases: $\mathcal{X} = [-2, 2]$, $\mathcal{X} = (-2, 2]$, $\mathcal{X} = (-2, 2)$, and $\mathcal{X} = (-3, 2]$.

(b) Let A be a bounded subset of $\mathbb{R}^d$. Identify the values of $\inf_x f(x), \sup_x f(x), \operatorname{argmin}_x f(x)$, and $\operatorname{argmax}_x f(x)$ for the function $f : \mathbb{R}^d \to \mathbb{R}$ defined by

$$f(x) = \inf_{y \in A} ||x - y||.$$

10. This problem shows how you can obtain inequalities for $\log(1 + x)$ and $\log(1 - x)$ from Taylor's theorem.

a. Expand the function $h(v) = \log v$ in a third order Taylor series around the point $v = 1$. (Thus you will be expressing $h(1 + x)$ in terms of x, $h(1)$, $h^{'}(1)$, $h^{''}(1)$, and $h^{'''}(u)$ for some u between 1 and $1 + x$. Note that x may be negative.)

b. By examining the final term in the series show that $\log(1 + x) \geq x - x^2/2$ for $x \geq 0$.

c. By examining the final term in the series show that $\log(1 - x) \leq -x - x^2/2$ for $0 \leq x < 1$.