STOR 565 Homework

1. (Operations on convex functions that produce new convex functions.) Let $C \subseteq \mathbb{R}^d$ be a convex set and let $f_1, \ldots, f_n : C \to \mathbb{R}$ be convex functions. Use the definition of convexity to establish the following.

   a. If $a_1, \ldots, a_n$ are non-negative then $g(x) = \sum_{i=1}^n a_i \, f_i(x)$ is convex on C.

   b. The function $g(x) = \max_{1 \leq i \leq n} f_i(x)$ is convex on C.

   c. If $h : \mathbb{R} \to \mathbb{R}$ is convex and increasing then $g(x) = h(f(x))$ is convex on C. (Recall that h is increasing if $u \leq v$ implies $h(u) \leq h(v)$).

2. Use the second derivative condition to establish whether the following functions are convex or concave. In each case, sketch the function.

   a. The function $f(x) = e^x$ on $(-\infty, \infty)$.

   b. The function $f(x) = \sqrt{x}$ on $(0, \infty)$.

   c. The function $f(x) = 1/x$ on $(0, \infty)$.

   d. The function $f(x) = \log x$ is on $(0, \infty)$.

3. Define the function $f(x) = x \log x$ for $x \in (0, \infty)$

   a. Sketch the function $f(x)$ and show that it is convex.

   b. Find the minimum and argmin of $f(x)$.

   b. Let $X > 0$ be a random variable. What can you say about the relationship between $\mathbb{E}(X \log X)$ and $\mathbb{E}X \log \mathbb{E}X$?

4. Let P be a probability measure on a set $\mathcal{X}$. Recall that if A and B are subsets of $\mathcal{X}$ and $P(B) > 0$, then the conditional probability of A given B is defined by

$$P(A \,|\, B) \;=\; \frac{P(A \cap B)}{P(B)}$$

Show the following.

a. If A and B are disjoint then $P(A \cup B \,|\, C) = P(A \,|\, C) + P(B \,|\, C)$

b. $P(A^c \,|\, B) = 1 - P(A \,|\, B)$

c. If $A \subseteq B$ then $P(A \,|\, C) \leq P(B \,|\, C)$

5. Let $\mathcal{X}$ be a set and let A, B be subsets of $\mathcal{X}$. Recall that the indicator function of A is defined by

$$\mathbb{I}_A(x) \;=\; \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in A^c \end{cases}$$

a. Show that $\mathbb{I}_{A^c} = 1 - \mathbb{I}_A$.

b. Show that $\mathbb{I}_A - \mathbb{I}_B = \mathbb{I}_{B^c} - \mathbb{I}_{A^c}$.

c. Show that $\mathbb{I}_{A \cap B} = \mathbb{I}_A \, \mathbb{I}_B$.

d. Let $u, v \in \{0, 1\}$. Show that $\mathbb{I}(u \neq v) = |\mathbb{I}(u = 1) - \mathbb{I}(v = 1)|$. Hint: Consider separately the cases $\mathbb{I}(u \neq v) = 0$ and $\mathbb{I}(u \neq v) = 1$.

6. Let (X, Y) be a predictor-response pair with $X \in \mathcal{X}$ and $Y \in \{0, 1\}$. Suppose that $\mathcal{X}$ is finite and that (X, Y) has joint probability mass function $p(x, y)$.

a. Express the prior probabilities $\pi_0 = P(Y = 0)$ and $\pi_1 = P(Y = 1)$ in terms of $p(x, y)$

b. Express the class conditional pmf $p_0(x)$ of X given $Y = 0$ in terms of $p(x, y)$ and π

c. Express the class conditional pmf $p_1(x)$ of X given $Y = 1$ in terms of $p(x, y)$ and π

d. Show that the marginal pmf of X can be written as $p(x) = \pi_0 \, p_0(x) + \pi_1 \, p_1(x)$

e. Use Bayes rule to show that $\eta(x) := P(Y = 1 \,|\, X = x) = \pi_1 p_1(x) / p(x)$

7. Let (X, Y) be a discrete random pair with joint probability mass function $p(x, y)$. Recall that $\mathbb{E}(Y|X) = \varphi(X)$ where $\varphi(x) = \sum_y y\, p(y|x)$. Establish the following.

a. If $Y \geq 0$ then $\mathbb{E}(Y|X) \geq 0$

b. $\mathbb{E}(aY + b|X) = a\, \mathbb{E}(Y|X) + b$

c. $\mathbb{E}\{\, \mathbb{E}(Y|X)\,\} = \mathbb{E}Y$

8. Let X, Y be non-negative random variables with joint density function $f(x, y) = y^{-1}\, e^{-x/y}\, e^{-y}$ for $x, y \geq 0$.

a. Find the marginal density $f(y)$ of Y

b. Find the conditional density $f(x \mid y)$ of X given $Y = y$

c. Find $\mathbb{E}[X \mid Y = y]$

d. Find $\mathbb{E}[X \mid Y]$