STOR 565 Homework 1

1. Let $X > 0$ be a positive, continuous random variable with density f_X. Use the CDF method to find the density of $Y = X^{-1}$ in terms of f_X.

2. Recall that the variance of a random variable X is defined by $\text{Var}(X) = \mathbb{E}(X - \mathbb{E}X)^2$. Carefully establish the following.

(a) If a, b are constants, then $\text{Var}(aX + b) = a^2 \, \text{Var}(X)$

(b) $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2$ (expand the square in the definition)

(c) $\mathbb{E}X^2 \geq (EX)^2$.

3. Let X be a random variable taking values in the finite interval $[0, c]$. You may assume that X is discrete, though this is not necessary for this problem.

(a) Show that $\mathbb{E}X \leq c$ and $\mathbb{E}X^2 \leq c \, \mathbb{E}X$.

(b) Recall that $\text{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$. Use the inequalities above to show that

$$\text{Var}(X) \;\leq\; c^2[u(1-u)] \quad \text{where} \quad u \;=\; \frac{\mathbb{E}X}{c} \in [0,1].$$

(c) Use the result of part (b) and simple calculus to show that $\text{Var}(X) \leq c^2/4$.

(d) Use the result in (c) to bound the variance of a random variable X taking values in an interval $[a, b]$ with $-\infty < a < b < \infty$.

4. Let $\phi(x)$ and $\Phi(x)$ be the density function and cumulative distribution function, respectively, of the standard normal distribution. Here we will find a useful upper bound on $1 - \Phi(x)$, which is the probability that a standard normal random variable exceeds x.

(a) Write down the formula for the density $\phi(t)$, and compute the derivative $\phi'(t)$.

(b) Justify the following sequence of equalities: For $x > 0$,

$$1 - \Phi(x) \;=\; \Phi(-x) \;=\; \int_{-\infty}^{-x} \phi(t)\, dt \;=\; \int_{-\infty}^{-x} \frac{1}{t} \cdot t\, \phi(t)\, dt.$$

(c) Integrate the last term above by parts to establish the useful inequality $1 - \Phi(x) \leq x^{-1}\, \phi(x)$ for $x > 0$.

5. Show that if $f(x)$ is bounded and $X \sim \text{Poiss}(\lambda)$ then $\mathbb{E}[\lambda f(X+1)] = \mathbb{E}[Xf(X)]$. Here $\text{Poiss}(\lambda)$ denotes the usual Poisson distribution with parameter λ.

6. Let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ be two sequences of real numbers.

(a) Show that $\min\{a_i\} + \min\{b_i\} \;\le\; \min\{a_i + b_i\} \;\le\; \min\{a_i\} + \max\{b_i\}$.

(Note that the first term is what you get if you can minimize over a_i and b_i separately and then add the results; the second term is what you get if you minimize the sum $a_i + b_i$, where a_i is always paired with b_i.)

(b) Show that $\max\{-b_i\} = -\min\{b_i\}$. (Use the fact that $a \le b$ if and only if $-b \le -a$.)

(c) Use (a) and (b) to show that

$$\min\{a_i\} - \max\{b_i\} \;\le\; \min\{a_i - b_i\} \;\le\; \min\{a_i\} - \min\{b_i\}.$$

7. By graphing the functions $f(x) = 1+x$ and $g(x) = e^x$, argue informally that $1+x \le e^x$ for every number x, and find one value of x where equality holds. Deduce from this inequality that $\log y \le y - 1$ for every $y > 0$.

8. The probability that an individual has a certain rate disease is about 1 percent. If they have the disease, the chance that they test positive is 90 percent. If they do not have the disease, the chance that they nevertheless test positive is 9 percent. What is the probability that someone who tests positive actually has the disease? (Use Bayes Formula.) What does this say about the test?