STOR 565 Homework

Show all work. Note: all logarithms are natural logarithms.

1. The empirical cumulative distribution function (CDF) of a sample $x = x_1, \ldots, x_m$ is defined by

$$F_x(t) \;=\; m^{-1} \sum_{i=1}^{m} \mathbb{I}(x_i \leq t)$$

The sum in the definition counts the number of data points that are less than or equal to t, so $F_x(t)$ is the fraction of data points that are less then or equal to t.

Suppose that x has four points: -3, -1, -1, and 5.

a. Find the following values of the empirical CDF by using the formula above: $F_x(-4)$, $F_x(0)$, $F_x(-1)$, $F_x(6)$

b. Sketch the empirical CDF for this data set as a function of t.

c. For what values of t is $F_x(t) = 0$?

d. For what values of t is $F_x(t) = 1$?

2. If $f : \mathcal{X} \to \mathbb{R}$ is a real-valued function then

$$\operatorname*{argmax}_{x \in \mathcal{X}} f(x) = \left\{ x \in \mathcal{X} : f(x) = \max_{u \in \mathcal{X}} f(u) \right\}.$$

is the set of points in x in $\mathcal{X}$ at which f is maximized. (Note that this is different from the maximum value of $f(x)$.) The argmin of f is similarly defined as the set of points in $\mathcal{X}$ where f is minimized.

(a) Identify the value of

$$\max_{x \in \mathcal{X}} x^2 \quad \text{and} \quad \operatorname*{argmax}_{x \in \mathcal{X}} x^2$$

in each of the following cases: $\mathcal{X} = [-2, 2]$, $\mathcal{X} = (-2, 2]$, and $\mathcal{X} = [-3, 0]$.

(b) Identify the min, argmin, max, and argmax of $f(x) = \sin(x)$ over $\mathcal{X} = [0, 2\pi]$.

(c) Let $x_1, \ldots, x_n \in \mathbb{R}$ be a data set. Identify the following

$$\operatorname*{argmin}_{a \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} (x_i - a)^2 \quad \text{and} \quad \min_{a \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} (x_i - a)^2$$

You may use results from the previous homework assignment.

3. Show that if $\mathbf{v}_1, \mathbf{v}_2$ are eigenvectors of a symmetric matrix $\mathbf{A}$ having different eigenvalues, then $\mathbf{v}_1, \mathbf{v}_2$ are orthogonal. Hint: Begin by taking transposes to show that $\mathbf{v}_1^t \mathbf{A} \mathbf{v}_2$ and $\mathbf{v}_2^t \mathbf{A} \mathbf{v}_1$ are equal; then use the definition of an eigenvector to simplify.

4. Let $\mathbf{A}$ be an $n \times n$ matrix. Show that if $\mathbf{A}$ has rank n then $\mathbf{A}\mathbf{x} = 0$ if and only if $\mathbf{x} = 0$. Hint: If $\mathbf{A}$ has rank n then its columns are linearly independent.

5. Let $\mathbf{A}$ and $\mathbf{B}$ be invertible $n \times n$ matrices. Argue that $(\mathbf{A}\,\mathbf{B})^{-1} = \mathbf{B}^{-1}\,\mathbf{A}^{-1}$.

6. Recall that the trace of an $n \times n$ matrix $\mathbf{A} = \{a_{ij}\}$ is the sum of its diagonal elements, that is $\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^n a_{ii}$.

a. Show that $\mathrm{tr}(\mathbf{A}) = \mathrm{tr}(\mathbf{A}^t)$.

b. Note that $(\mathbf{A}\,\mathbf{B})_{ii} = \sum_{j=1}^n a_{ij}\, b_{ji}$ (Why?). Use this to show that $\mathrm{tr}(\mathbf{A}\,\mathbf{B}) = \mathrm{tr}(\mathbf{B}\,\mathbf{A})$.

c. By applying the identity of part b. multiple times, show that

$$\mathrm{tr}(\mathbf{A}\,\mathbf{B}\,\mathbf{C}) \;=\; \mathrm{tr}(\mathbf{B}\,\mathbf{C}\,\mathbf{A}) \;=\; \mathrm{tr}(\mathbf{C}\,\mathbf{A}\,\mathbf{B})$$

d. Suppose that $\mathbf{B} = \{b_{ij}\}$ is an $m \times n$ matrix. By considering $(\mathbf{B}^t\,\mathbf{B})_{ii}$, show that

$$\mathrm{tr}(\mathbf{B}^t\,\mathbf{B}) \;=\; \sum_{i=1}^m \sum_{j=1}^n b_{ij}^2$$

7. Show that if $A \in \mathbb{R}^{n \times n}$ is non-negative definite then all its eigenvalues are non-negative. Hint: Apply the definition of non-negative definite to the eigenvectors of A.

8. Establish the following properties of the Frobenius norm for matrices.

(a) $||\mathbf{A}|| = 0$ if and only if $\mathbf{A} = 0$

(b) $||b\mathbf{A}|| = |b|\,||\mathbf{A}||$

(c) If $\mathbf{A} \in \mathbb{R}^{m \times n}$ then $||\mathbf{A}||^2 = \sum_{i=1}^m ||a_{i\cdot}||^2 = \sum_{j=1}^n ||a_{\cdot j}||^2$

(d) $||\mathbf{A}\mathbf{B}|| \leq ||\mathbf{A}||\,||\mathbf{B}||$. Hint: Use Cauchy-Schwarz.

9. Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are orthogonal vectors in $\mathbb{R}^n$. Show that $||\sum_{i=1}^k \mathbf{v}_i||^2 = \sum_{i=1}^k ||\mathbf{v}_i||^2$. Interpret this in terms of the Pythagorean formula relating the length of the hypotenuse of a right triangle to the lengths of the other edges.