# STOR 654 Homework 5

1. Let $\mathcal{P} = \{f(x|\theta) : \theta \in \Theta\}$ be a family of distributions on a sample space $\mathcal{X}$. Suppose that $T : \mathcal{X} \to \mathcal{T}$ is sufficient for $\mathcal{P}$ and that $S : \mathcal{S} \to \mathcal{S}$ is another statistic such that $S = \phi(T)$ for some bijection $\phi : \mathcal{T} \to \mathcal{T}$. Show that $S$ is sufficient for $\mathcal{P}$.

2. Show that for $a, b > 0$ we have $\sqrt{a/2} + \sqrt{b/2} \leq \sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$.

3. Let $\mathcal{P}$ be an exponential family generated by $T : \mathcal{X} \to \mathbb{R}^d$ and $h(x) : \mathcal{X} \to [0, \infty)$ with parameters $\eta \in \mathbb{R}^d$. Show that the maximum likelihood estimator of $\eta$ coincides with the method of moments estimator.

4. (Using symmetry) Let $X_1, \cdots, X_n$ be iid positive random variables such that $\mathbb{E}X_k = a$ and $\mathbb{E}X_k^{-1} = b$ are finite. For $1 \leq k \leq n$, let $S_k = X_1 + \cdots + X_k$ and let $V_k = X_k/S_n$.

   (a) Show that $\mathbb{E}S_m^{-1} < \infty$ for $m = 1, \cdots, n$.

   (b) Argue informally that $V_1, \cdots, V_n$ have the same distribution.

   (c) Conclude from part (b) that $\mathbb{E}(S_m/S_n) = m/n$ for $1 \leq m \leq n$.

   (d) Show that $\mathbb{E}(S_n/S_m) = 1 + (n-m)\, a\, \mathbb{E}S_m^{-1}$ for $1 \leq m \leq n$.

   (e) Verify the inequality $x + x^{-1} \geq 2$ for $x > 0$.

   (f) Use the inequality of part (e) with $x = c(S_m/S_n)$, and an appropriate constant $c$, to show that $\mathbb{E}(S_n/S_m) \geq n/m$ for $1 \leq m \leq n$.