STOR 655 Homework 9

1. Let $X_1, \ldots, X_n \in \mathcal{X}$ be i.i.d. and let $\mathcal{G}$ be a family of function $g : \mathcal{X} \to [-c, c]$. Define

$$f(x_1^n) \;=\; \sup_{g \in \mathcal{G}} \left| n^{-1} \sum_{i=1}^{n} g(x_i) \;-\; \mathbb{E} g(X) \right|$$

Find the difference coefficients $c_1, \ldots, c_n$ of f, and use these to establish concentration bounds for the random variable $f(X_1^n)$.

2. Let $X \sim \mathcal{N}_n(0, I)$ and $Y \sim \mathcal{N}_n(0, I)$ be independent multinormal random variables. For $0 \leq \theta \leq \pi/2$ define random vectors

$$X(\theta) \;=\; X \sin\theta \;+\; Y \cos\theta$$

$$\dot{X}(\theta) \;=\; X \cos\theta \;-\; Y \sin\theta$$

(a) Show that for each θ, $X(\theta)$ and $\dot{X}(\theta)$ have the same distribution as X.

(b) Show that for each θ, $X(\theta)$ and $\dot{X}(\theta)$ are independent.

3. *Concentration for norms of Gaussian random vectors.* Let $Y \sim \mathcal{N}_d(0, \Sigma)$ and consider the random variable $U = ||Y||$.

(a) Show that $U = F(X)$ where $X \sim \mathcal{N}_d(0, I)$ and $F(x) = ||\Sigma^{1/2} x||$

(b) Show that F Lipschitz with constant

$$L \;\leq\; \sup_{u \in \mathbb{R}^d} \frac{||\Sigma^{1/2} u||}{||u||}$$

(c) Find a bound on the right hand side of the inequality above involving the largest eigenvalue of Σ.

(d) Find a concentration inequality for U.

4. Let $X_1, \ldots, X_n \in \mathbb{R}^d$ be independent random vectors such that $\mathbb{E} X_i = 0$ and $||X_i|| \leq c_i/2$ with probability one, where $||u|| = (u^t u)^{1/2}$ is the ordinary Euclidean norm. Let $\alpha = (1/4) \sum_{i=1}^n c_i^2$.

(a) Show that $\mathbb{E} || \sum_{i=1}^n X_i || \leq \sqrt{\alpha}$.

(b) Use the bounded difference inequality and the inequality in part (a) to show that for all $t \geq \alpha$

$$P\left(||\sum_{i=1}^{n} X_i|| > t\right) \;\leq\; \exp\left\{\frac{(t-\sqrt{\alpha})^2}{2\alpha}\right\}$$

5. Let $a_1, \ldots, a_n$ be real numbers. Show that $n^{-1}\sum_{k=1}^{n}|a_k| \leq (n^{-1}\sum_{k=1}^{n} a_k^2)^{1/2}$.

6. Let $\Gamma(x)$ be the standard Gamma function, defined for $x > 0$. Show that if $Z \sim \mathcal{N}(0,1)$ then for each $p \geq 1$

$$\mathbb{E}|Z|^p \;=\; \frac{2^{p/2}}{\sqrt{\pi}}\Gamma((1+p)/2)$$

Deduce from this fact and Stirling's approximation that $||Z||_p := (\mathbb{E}|Z|^p)^{1/p} = O(p^{1/2})$.

7. Let X be a random variable satisfying the concentration type inequality $\mathbb{P}(|X| > t) \leq a\, e^{-b\,t^2}$ for all $t \geq 0$. Show that

$$\mathbb{E}|X| \leq \sqrt{\frac{1+\log a}{b}}.$$

Hint: Note that for $s \geq 0$ we have $\mathbb{E}X^2 \leq s + \int_s^\infty \mathbb{P}(X^2 \geq t)$. Use Cauchy-Schwartz.

8. Let $X_1, \ldots, X_n$ be random variables with moment generating functions $\varphi_{X_i}(s) \leq \varphi(s)$ for each $s \geq 0$.

(a) Using the argument in class for Gaussian random variables, show that

$$\mathbb{E}\max(X_1, \ldots, X_n) \;\leq\; \inf_{s>0} \frac{\log n + \log \varphi(s)}{s}.$$

Suppose now that $U_1, \ldots, U_n$ are Gamma(α, β) random variables.

(b) Show that the moment generating function of U_i is $\varphi(s) = (1 - s\,\beta)^{-\alpha}$.

(c) Using the bound from part (a) and an appropriate choice of s, which can be found by inspection, show that

$$\mathbb{E}\max(U_1, \ldots, U_n) \;\leq\; \frac{2\,\beta\,\log n}{1 - n^{-1/\alpha}}.$$