STOR 655 Homework 10

1. Let $X_1, \ldots, X_n$ be independent standard normal random variables. Here we identify upper and lower bounds for the expectation of $K_n := \max_{1 \le i \le n} |X_i|$.

   (a) Using the bound from class and the fact that $K_n = \max_i(X_i, -X_i)$ show that $\mathbb{E}K_n \le (2 \log 2n)^{1/2}$.

   (b) Let $\Phi()$ be the CDF of the standard normal. Show that

   $$K_n \;=\; \Phi^{-1}\left(\frac{1}{2} + \frac{1}{2}\max_{1 \le i \le n} V_i\right)$$

   where $V_1, \ldots, V_n$ are independent Uniform$(0,1)$ random variables.

   (c) Show that $\Phi^{-1}(u)$ is convex on $[1/2, 1)$. Apply Jensen's inequality to the expression in (b) to obtain the bound $\mathbb{E}K_n \ge \Phi^{-1}\left(1 - 1/(2n+2)\right)$.

   (d) Show that $\Phi^{-1}(1 - t^{-1})/(2 \log t)^{1/2} \to 1$ as $t \to \infty$.

   (e) Conclude from (a), (c), and (d) that $\mathbb{E}K_n/(2 \log n)^{1/2} \to 1$ as $n \to \infty$.

2. *Extreme value theory for the Gaussian.* Let a_n and b_n be the extreme value scaling and centering constants for the maximum M_n of n independent standard Gaussian random variables.

   (a) Fix $x \in \mathbb{R}$ and let $x_n = x/a_n + b_n$. Show that $n\,\phi(x_n)/x_n \to e^{-x}$ as n tends to infinity. [In your calculations, identify and pay careful attention to the leading order terms.]

   (b) Using the result of part (a) and the standard Gaussian tail bound from an earlier homework, show that $n(1 - \Phi(x_n)) \to e^{-x}$.

   (c) Use part (b) and the lemma from lecture to show that as n tends to infinity

   $$\mathbb{P}(a_n(M_n - b_n) \le x) \to G(x) = e^{-e^{-x}}$$

   (d) Show that $G(x)$ is the CDF of $-\log V$ where $V \sim \text{Exp}(1)$.

3. Let $U_1, \ldots, U_n$ be independent Uniform$(0, \theta)$ random variables. Find $\mathbb{E}\left[\max_{1 \le j \le n} U_j\right]$.

4. Let $\{C_\lambda : \lambda \in \Lambda\}$ be convex sets. Show that the intersection $C = \cap_{\lambda \in \Lambda} C_\lambda$ is convex.

5. Show that the following subsets of $\mathbb{R}^d$ are convex.

a. The emptyset

b. The hyperplane $H = \{x : x^t u = b\}$

c. The halfspace $H_+ = \{x : x^t u > b\}$

d. The ball $B(x_0, r) = \{x : ||x - x_0|| \leq r\}$

6. Show that if $f_1, \ldots, f_k$ are convex functions defined on the same set, and $w_1, \ldots, w_k$ are non-negative, then $f = \sum_{j=1}^k w_j f_j$ is convex.

7. Let $\{f_\lambda : \lambda \in \Lambda\}$ be convex functions defined on a common set C. Show that the supremum $f = \sup_{\lambda \in \Lambda} f_\lambda$ is convex.

8. Establish the following facts about the Gaussian mean width $w(K)$ of a bounded set $K \subseteq \mathbb{R}^n$.

(a) If $K_1 \subseteq K_2$ then $w(K_1) \leq w(K_2)$

(b) $w(K) \geq 0$

(c) If $A \in \mathbb{R}^{n \times n}$ is orthogonal then $w(AK) = w(A)$

(d) For each $u \in \mathbb{R}^n$, $w(K + u) = w(K)$

(e) $w(K) = w(\text{conv}(K))$

(f) $\sqrt{2/\pi}\, \text{diam}(K) \leq w(K) \leq n^{1/2}\, \text{diam}(K)$

(g) $w(K) \leq 2\, \mathbb{E} \sup_{x \in K} \langle x, V \rangle$ with $V \sim \mathcal{N}_n(0, I)$

9. Recall that the convex hull of a set $A \subseteq \mathbb{R}^d$, denoted $\text{conv}(A)$, is the intersection of all convex sets C containing A. Show that $\text{conv}(A)$ is equal to the set of all convex combinations $\sum_{i=1}^k \alpha_i x_i$, where $k \geq 1$ is finite, $x_1, \ldots, x_k \in A$, and the coefficients α_i are non-negative and sum to one.