GAUSSIAN COMPARISON LEMMA

1. Gaussian Comparision Lemma

**Lemma 1.1.** Let $G : \mathbb{R}^n \to \mathbb{R}$ be a bounded, twice continuously differentiable function with bounded derivatives

$$G_i(x) = \frac{\partial G(x)}{\partial x_i} \quad 1 \leq i \leq n \quad \text{and} \quad G_{ij} = \frac{\partial^2 G(x)}{\partial x_i \partial x_j} \quad 1 \leq i, j \leq n.$$ 

If $X \sim N_n(0, \Sigma_X)$ and $Y \sim N_n(0, \Sigma_Y)$ are normal random vectors then

$$E G(Y) - E G(X) = \frac{1}{2} \sum_{i,j=1}^{n} \Delta_{ij} \int_0^1 E G_{ij}(X^t) \, dt$$

where $\Delta_{ij} = E Y_i Y_j - E X_i X_j = (\Sigma_Y - \Sigma_X)_{ij}$ and $X^t \sim N_n(0, \Sigma_t)$ with $\Sigma_t := (1 - t) \Sigma_X + t \Sigma_Y$.

**Proof:** Assume without loss of generality that $X$ and $Y$ are independent. For each $t \in [0, 1]$ define the random vector

$$X^t = (1 - t)^{1/2} X + t^{1/2} Y$$

and the associated function $\varphi(t) = E G(X^t)$. Note that $X^0 = X$, $X^1 = Y$, and that $X^t \sim N_n(0, \Sigma_t)$, where $\Sigma_t$ is defined as in the statement of the lemma. Thus

$$E G(Y) - E G(X) = \varphi(1) - \varphi(0) = \int_0^1 \varphi'(t) \, dt,$$

and it suffices to show that for each $t \in (0, 1)$

$$\varphi'(t) = \frac{1}{2} \sum_{i,j=1}^{n} \Delta_{ij} E G_{ij}(X^t). \quad (1.1)$$

To this end, fix $t \in (0, 1)$ and note that $X^t$ is distributed as $\Sigma_t^{1/2} Z$ where $Z \sim N(0, I)$ is a standard normal random vector with independent components. To simplify notation, let $A_t := \Sigma_t^{1/2}$. It follows from our regularity assumptions and the chain rule that

$$\varphi'(t) = \frac{d}{dt} E G(A_t Z) = E \left[ \frac{d}{dt} G(A_t Z) \right] = E \left[ \sum_{i=1}^{n} G_i(A_t Z) \frac{d}{dt} (A_t Z)_i \right]$$

where $A'_t$ denotes the entry-by-entry derivative of the matrix $A_t$. Fix $i, j$ for the moment and define the function

$$H_{ij}(s) := E G_i(A_t Z_s) \quad \text{where} \quad Z_s := (Z_1, \cdots, Z_{j-1}, s, Z_{j+1}, \cdots, Z_n). \quad (1.2)$$

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It follows from a simple conditioning argument and Gaussian integration by parts that

\[ E[Z_j G_i(A_t Z)] = E[Z_j H_{ij}(Z_j)] = E H'_{ij}(Z_j). \]  

By another application of the chain rule,

\[ H'_{ij}(s) = E \left[ \frac{d}{ds} G_i(A_t Z_s) \right] = \sum_{k=1}^n E \left[ G_{ik}(A_t Z_s) \frac{d}{dt}(A_t Z_s)_k \right] \]

\[ = \sum_{k=1}^n (A_t)_{jk} E G_{ik}(A_t Z_s). \]

Thus, as \( Z_1, \ldots, Z_n \) are independent,

\[ E H'_{ij}(Z_j) = \sum_{k=1}^n (A_t)_{jk} E G_{ik}(A_t Z). \]

Combining this last equation with (1.2), we find that

\[ \varphi'(t) = \sum_{i,k=1}^n E G_{ik}(A_t Z) \cdot \sum_{j=1}^n (A'_t)_{ij}(A_t)_{jk} \]

\[ = \sum_{i,k=1}^n E G_{ik}(X^t) \cdot (A'_t A_t)_{ik}. \]  

Recalling that \( A_t = \Sigma_t^{1/2} \), it is easy to see that \( (A^2_t)_{ik}' = (\Sigma_t)'_{ik} = \Delta_{ik} \). Furthermore, as \( A_t \) and \( A'_t \) are symmetric,

\[ (A^2_t)' = A'_t A_t + A_t A'_t = A'_t A_t + (A'_t A_t)^T. \]

Fix \( 1 \leq i < k \leq n \). Continuity of the second partial derivatives ensures that \( G_{ik} = G_{ki} \), and therefore

\[ E G_{ik}(X^t) \cdot (A'_t A_t)_{ik} + E G_{ki}(X^t) \cdot (A'_t A_t)_{ki} \]

\[ = E G_{ik}(X^t) \left( (A'_t A_t)_{ik} + (A'_t A_t)_{ki} \right) \]

\[ = E G_{ik}(X^t) (A^2_t)'_{ik} = E G_{ik}(X^t) \Delta_{ik}, \]

where the penultimate equality follows from (1.5). A similar argument shows that \( (A'_t A_t)_{ii} = \Delta_{ii}/2 \). Thus (1.1) follows from (1.2), and the proof is complete.
1.1. Further Reading: Gaussian Tail Bounds. Let \( \Phi(x) = 1 - \Phi(x) \) where \( \Phi(x) \) is the cumulative distribution function of the standard Gaussian distribution. Recall that for \( x > 0 \),

\[
\Phi(x) \leq \frac{1}{\sqrt{2\pi x}} e^{-x^2/2}.
\]  

The proof of Theorem 2 requires an inequality for the probability that two correlated Gaussian random variables each exceeds a common threshold.

**Lemma 1.2.** Let \((Z, Z_\rho)\) be jointly Gaussian random variables with mean 0, variance 1, and correlation \( \mathbb{E}(ZZ_\rho) = \rho \in (-1, 1) \). Then for any \( u > 0 \),

\[
P(Z > u, Z_\rho > u) \leq \frac{(1 + \rho)^2}{2\pi u^2 \sqrt{1 - \rho^2}} \exp\left(-u^2/(1 + \rho)\right).
\]

**Proof of Lemma 1.2.** Fix \( u > 0 \). When \( \rho \geq 0 \) the proof follows from known inequalities in the literature (see R. Willink, Bounds on the bivariate normal distribution function, Comm. Statist. Theory Methods 33 (2004), pp.2281-2297). Here we consider the case \( \rho < 0 \). Note that we may write \( Z_\rho = \rho Z + \sqrt{1 - \rho^2} Z' \), where \( Z' \) is a standard Gaussian random variable independent of \( Z \). By conditioning on the value of \( Z \), it is easy to see that

\[
P(Z > u, Z_\rho > u) = \int_u^\infty \Phi(g(t)) \phi(t) \, dt \quad \text{where} \quad g(t) = \frac{u - \rho t}{\sqrt{1 - \rho^2}}.
\]

Now define

\[
\eta = \sqrt{\frac{1 - \rho}{1 + \rho}} \quad \text{and} \quad h(x) = e^{x^2/2} \Phi(x).
\]

As \( h'(x) = xe^{x^2/2} \Phi(x) - 1/\sqrt{2\pi} \), inequality (1.6) implies that \( h(x) \) is decreasing for \( x > 0 \). It follows from equation (1.8) that

\[
P(Z > u, Z_\rho > u) = \int_u^\infty e^{-g(t)^2/2} h(g(t)) \phi(t) \, dt
\]

\[
\leq h(g(u)) \int_u^\infty e^{-g(t)^2/2} \phi(t) \, dt
\]

\[
= h(\eta u) \int_u^\infty e^{-g(t)^2/2} \phi(t) \, dt,
\]

where in the last step we have used the fact that \( g(u) = \eta u \). Routine algebra and a change of variables establishes that

\[
\int_u^\infty e^{-g(t)^2/2} \phi(t) \, dt = e^{-u^2/2} \int_u^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(t - \rho u)^2}{2(1 - \rho^2)}\right) \, dt
\]

\[
= \sqrt{1 - \rho^2} e^{-u^2/2} \Phi(\eta u).
\]

Combining (1.9), (1.11), and inequality (1.6) yields the bound (1.7), as desired.