STOR 655 Homework 4

1. Let $X_1, X_2, \ldots \in \mathbb{R}^d$ be random vectors, possibly defined on different probability spaces, such that $X_n \Rightarrow c$ where $c \in \mathbb{R}^d$ is constant. Show that $X_n \to c$ in probability. Hint: Note that for $\delta > 0$, $I(||x - c|| > \delta) \leq f_\delta(x)$ where $f_\delta(x) = \delta^{-1}||x - c|| \wedge 1$.

2. Find the moment generating function $\psi(t) = \mathbb{E}e^{tZ}$ of a standard normal random variable. Use the series expansion of $\psi(t)$ to find the moments EZ^{2k} for $k \geq 1$.

3. Give a simple example of a family of functions $g_n : \mathbb{R} \to [0, 1]$ such that $g_n(x) \to g(x) = 1$ for each $x \in \mathbb{R}$ but $\sup_{x \in \mathbb{R}} |g_n(x) - g(x)| = 1$ for each n. *Optional:* Find an example like that above with functions $g_n : [0, 1] \to [0, 1]$.

4. Explain and prove the relation $o_p(O_p(1)) = o_p(1)$ for random variables.

5. Let $\Phi(x)$ and $\phi(x)$ be the cumulative distribution function and density, respectively, of the standard normal distribution. In this problem, you are asked to find a useful approximation to $1 - \Phi(x)$ when x is large. Note that for $x > 0$,

$$1 - \Phi(x) \;=\; \Phi(-x) \;=\; \int_{-\infty}^{-x} \frac{1}{t} \cdot t\,\phi(t)\,dt$$

(a) Apply integration-by-parts to the last integral above. Use the resulting expression establish the upper bound $1 - \Phi(x) \leq x^{-1}\,\phi(x)$ for $x > 0$.

(b) Apply the same steps to the integral appearing in the integration-by-parts. Use this to establish the lower bound

$$1 - \Phi(x) \geq \left(\frac{1}{x} - \frac{1}{x^3}\right)\phi(x) \quad \text{for} \quad x > 0.$$

(c) Conclude that as $x \to \infty$ $(1 - \Phi(x)) = \frac{\phi(x)}{x}(1 + o(1))$