STOR 655 Homework 1
Final Version

1. Show that $(1 + u/3)^3 \geq 1 + u$ for every $u \geq 0$.

2. Let $h(u) = (1 + u)\log(1 + u) - u$. (This function appears in Bennett's exponential inequality for sums of independent, bounded random variables.)

(a) By considering the first few terms of Taylor expansion of the function $h(\cdot)$ around zero, show that for every $u \geq 0$

$$h(u) \geq \frac{u^2}{2 + 2u}$$

(b) (Optional) Use calculus to establish the stronger bound that for every $u \geq 0$

$$h(u) \geq \frac{u^2}{2 + 2u/3}$$

3. Let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ be two sequences of numbers.

(a) Show that $\min\{a_i\} + \min\{b_i\} \ \leq \ \min\{a_i + b_i\} \ \leq \ \min\{a_i\} + \max\{b_i\}$

(b) Show that $-\min\{a_i\} = \max\{-a_i\}$ and $-\max\{a_i\} = \min\{-a_i\}$. Use these relations in conjunction with the results of part (a) to get a related chain of inequalities involving maxima.

(c) Show that $\max\{a_i\} - \max\{b_i\} \ \leq \ \max\{|a_i - b_i|\}$

4. Establish the following relations for random vectors X and Y of appropriate dimension.

(a) $\mathbb{E}(AX) = A\mathbb{E}X$

(b) $\mathrm{Var}(AX) = A\,\mathrm{Var}(X)A^t$

(c) $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)^t$

(d) $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + \mathrm{Cov}(X, Y) + \mathrm{Cov}(Y, X)$

(e) If X, Y are independent, then $\mathrm{Cov}(X, Y) = 0$

5. Let $U \sim \mathcal{N}_d(\mu, \Sigma)$ and let $V = \Sigma^{1/2} Y + \mu$ where $Y \sim \mathcal{N}_d(0, I)$.

(a) Show that $\mathbb{E}U = \mathbb{E}V$ and that $\text{Var}(U) = \text{Var}(V)$.

(b) Fix $v \in \mathbb{R}^d$. Find the distributions of the random variables $v^t U$ and $v^t V$. Note that these distributions are the same.

6. Let $x = (x_1, \ldots, x_d)^t \in \mathbb{R}^d$ and let $||x||$ be the Euclidean (ℓ_2) norm of x. Show that for $1 \leq i \leq d$,

$$|x_i| \;\leq\; ||x|| \;\leq\; |x_1| + \cdots + |x_d|.$$

Use the inequalities to show that if $X \in \mathbb{R}^d$ is a random vector then $\mathbb{E}||X|| < \infty$ if and only if $\mathbb{E}|X_i| < \infty$ for $1 \leq i \leq d$.

7. Give a simple example of random vectors $X, Y \in \mathbb{R}^2$ such that $\text{Cov}(X, Y) \neq \text{Cov}(Y, X)$.

8. Let $X \sim \mathcal{N}(0, \sigma^2)$. Establish the identity

$$\mathbb{E} \exp\{aX^2 + bX\} \;=\; \frac{1}{\sqrt{1 - 2a\sigma^2}} \exp\left\{ \frac{\sigma^2 b^2}{2(1 - 2a\sigma^2)} \right\}$$

Hint: Write the expectation as an integral. Combine terms in the exponent and complete the square. Remove the constant factor and perform a simple change of variables to evaluate the remaining integral.