STOR 565 Homework

1. Define what it means for a set $C \subseteq \mathbb{R}^d$ to be convex. Let $w \in \mathbb{R}^d$ be a vector and $b \in \mathbb{R}$ a constant. Show that $C = \{x : w^t x \geq b\}$ and $D = \{x : w^t x = b\}$ are convex subsets of $\mathbb{R}^d$.

2. Let $C_1, \ldots, C_n \subseteq \mathbb{R}^d$ be convex. Show that the intersection $\cap_{i=1}^n C_i$ is convex.

3. Use the second derivative condition to establish whether the following functions are convex or concave. In each case, sketch the function.

   a. The function $f(x) = e^x$ on $(-\infty, \infty)$.

   b. The function $f(x) = \sqrt{x}$ on $(0, \infty)$.

   c. The function $f(x) = 1/x$ on $(0, \infty)$.

   d. The function $f(x) = \log x$ on $(0, \infty)$.

Now let $X > 0$ be a positive random variable. Write out the conclusion of Jensen's inequality for each of the functions above.

4. (Operations on convex functions that produce new convex functions) Let $C \subseteq \mathbb{R}^d$ be a convex set and let $f_1, \ldots, f_n : C \to \mathbb{R}$ be convex functions. Use the definition of convexity to establish the following.

   a. If $a_1, \ldots, a_n$ are non-negative then $g(x) = \sum_{i=1}^n a_i \, f_i(x)$ is convex on C.

   b. The function $g(x) = \max_{1 \leq i \leq n} f_i(x)$ is convex on C.

   c. If $h : \mathbb{R} \to \mathbb{R}$ is convex and increasing then $g(x) = h(f(x))$ is convex on C. (Recall that h is increasing if $u \leq v$ implies $h(u) \leq h(v)$).

5. Define the function $f(x) = x \log x$ for $x \in (0, \infty)$

   a. Sketch the function $f(x)$ and show that it is convex.

   b. Find the minimum and argmin of $f(x)$.

   b. Let $X > 0$ be a random variable. What can you say about the relationship between $\mathbb{E}(X \log X)$ and $\mathbb{E}X \log \mathbb{E}X$?

6. Recall that the Frobenius norm of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ is given by $||A|| = \sqrt{\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2}$, the square root of the sum of the squares of the entries of the matrix. Establish the following properties of the Frobenius norm for matrices.

(a) $||\mathbf{A}|| = 0$ if and only if $\mathbf{A} = 0$

(b) $||b\mathbf{A}|| = |b|\,||\mathbf{A}||$

(c) $||\mathbf{A}||^2 = \sum_{i=1}^m ||a_{i\cdot}||^2 = \sum_{j=1}^n ||a_{\cdot j}||^2$. Here $a_{i\cdot}$ denotes the ith row of A, and $a_{\cdot j}$ denotes the jth column of A.

(d) $||\mathbf{AB}|| \leq ||\mathbf{A}||\,||\mathbf{B}||$. Hint: Use Cauchy-Schwarz.

7. Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are orthogonal vectors in $\mathbb{R}^n$. Show that $||\sum_{i=1}^k \mathbf{v}_i||^2 = \sum_{i=1}^k ||\mathbf{v}_i||^2$. Interpret this in terms of the Pythagorean formula relating the length of the hypotenuse of a right triangle to the lengths of the other edges.

8. Show that if $A \in \mathbb{R}^{n\times n}$ is non-negative definite then all its eigenvalues are non-negative. Hint: Apply the definition of non-negative definite to the eigenvectors of A.

9. Some general questions about rooted binary trees. Refer to the notes on clustering for the definition.

a. Draw a rooted binary tree with 3 nodes. How many leaves does it have? How many internal nodes does it have?

b. Draw a rooted binary tree with 5 nodes. How many leaves does it have? How many internal nodes does it have?

c. Draw essentially different rooted binary trees with 7 nodes. Do they have the same number of internal nodes? Do they have the same number of leaves?

d. Formulate a conjecture about the relationship between the number of internal nodes and the number of leaves in a rooted binary tree.

e. (Optional) Prove your conjecture using induction.