STOR 655 Homework 13

1. Show that if $U \sim \chi^2_n$ with $n \geq 3$ then $\mathbb{E}U^{-1} = 1/(n-2)$.

2. Show that if $Y \sim \mathcal{N}(0, \sigma^2)$ then $\mathbb{E}\{|Y| I(|Y| > c)\} \leq \sigma \exp\{-c^2/2\sigma^2\}$

3. Show that $xy \leq 3x^2 + y^2/3$ for $x, y \geq 0$.

4. Show that if $d \geq 3$ then $\int_{\mathbb{R}^d} ||u||^{-2}\, e^{-||u||^2} = c \int_0^\infty e^{-r^2} r^{d-3} dr < \infty$.

5. Let $X_1, \ldots, X_n$ be independent random variables with $X_i \sim \mathcal{N}_n(\theta_i, 1)$. Suppose that we wish to simultaneously test the hypotheses $\mathrm{H}_{0,i} : \theta_i = 0$ vs. $\mathrm{H}_{1,i} : \theta_i \neq 0$ for $1 \leq i \leq n$. Consider a simple threshold test in which we reject $\mathrm{H}_{0,i}$ if $|X_i| > \tau$ and accept $\mathrm{H}_{0,i}$ otherwise. Using the asymptotic results on Gaussian extreme values, find a value of the threshold τ, depending on n, so that the family-wise error rate of the test under the global null $\theta_1 = \cdots = \theta_n = 0$ is (approximately) controlled at 5%.

6. Recall that the L_p-norm of a random variable X is defined by $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$. Establish Lyapunov's inequality: If $p \leq q$ then $\|X\|_p \leq \|X\|_q$. [Hint: Apply Hölder's inequality with an appropriate choice of conjugate exponents to $|X|^p \cdot 1$.]

7. Use the general versions of Stein's Lemma given in class to show that if $Y \sim \mathcal{N}_n(\theta, I)$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ is a sufficiently nice function then $\mathbb{E}[(Y - \theta)^T g(Y)] = \mathbb{E}[\nabla^T g(Y)]$.

8. In class we established a risk bound for the James-Stein estimator for observations $Y \sim \mathcal{N}_n(\theta, I)$. By looking over the proof, establish an analogous bound in the case $Y \sim \mathcal{N}_n(\theta, \sigma^2 I)$ with $\sigma > 0$ known.

9. Show that $|e^a - e^b| \leq e^b\, e^{|a-b|}\, |a - b|$