# STOR 655 Homework 11

1. Identify the extreme points (if any) of the following convex sets.

   a. The hyperplane $H = \{x : x^t u = b\}$

   b. The halfspace $H_+ = \{x : x^t u > b\}$

   c. The closed ball $\overline{B}(x_0, r) = \{x : ||x - x_0|| \leq r\}$

2. Let $f : C \to \mathbb{R}$ be a strictly convex function defined on a convex set $C \subseteq \mathbb{R}^n$. Show that $\text{argmax}_{x \in C} f(x)$ is contained in the set of extreme points of $C$.

3. (Set sums and scaler products) Given sets $A, B \subseteq \mathbb{R}^d$ and a constant $\alpha \in \mathbb{R}$ define the set sum and set scaler product as follows:

$$A + B \;=\; \{x + y : x \in A \text{ and } y \in B\} \qquad \alpha A \;=\; \{\alpha x : x \in A\}$$

   a. (Optional) Show that if $A$ is open then $A + B$ is open regardless of whether $B$ is open.

   b. Show that if $A$ and $B$ are convex, then so is $A + B$.

   c. If $A$ is convex is $A + B$ necessarily convex?

   d. Show by example that, in general, $2A \neq A + A$.

   d. Show that if $A$ is convex then $\alpha A + \beta A = (\alpha + \beta)A$ for all $\alpha, \beta \geq 0$.

4. Let $f$ be a convex function defined on a convex set $C$. Show that for each $\alpha \in \mathbb{R}$ the level set $L(\alpha) = \{x : f(x) \leq \alpha\}$ is convex.

5. Let $X \in \mathbb{R}$ be an integrable random variable with CDF $F(x)$, and for $0 < p < 1$ let $h_p(x, \theta) = p(x - \theta)_+ + (1 - p)(\theta - x)_+$.

   (a) Show that for each fixed $p$ and $x$, $h_p(x, \theta)$ is a convex function of $\theta$.

   (b) Show that, under reasonable assumptions on $F$, the quantity $\mathbb{E}h_p(X, \theta)$ is minimized by the $p$th quantile $F^{-1}(p)$ of $X$. Clearly state any assumptions that you make.

   (c) What does the result of part (b) tell you in the special case $p = 1/2$.

6. Let $A(t) = \{A_{i,j}(t) : 1 \leq i, j \leq n\}$ be a matrix whose entries are differentiable functions of a real number $t$. Define the entry-wise derivative

$$A'(t) \;=\; \{A'_{i,j}(t) : 1 \leq i, j \leq n\}$$

(a) Show that the entry-wise derivate obeys the usual product rule, that is,

$$[A(t)B(t)]' \;=\; A(t)B'(t) \,+\, A'(t)B(t)$$

7. Show that $||x||_\infty = \lim_{p \nearrow \infty} ||x||_p$. For $0 \leq p \leq 1$ define $||x|_p = \sum_{i=1}^d |x_i|^p$. Show that $||x||_0 = \lim_{p \searrow 0} ||x||_p$.

8. Show that if $u, v \in \mathbb{R}^n$ are orthogonal then $||u||_2 + ||v||_2 \leq \sqrt{2}||u + v||_2$.

9. Let $f$ be a convex function on an open interval $I \subseteq \mathbb{R}$ and let $a < b < c$ be in $I$.

(a) Show that

$$\frac{f(b) - f(a)}{b - a} \;\leq\; \frac{f(c) - f(a)}{c - a} \;\leq\; \frac{f(c) - f(b)}{c - b}.$$

(Hint: express $b$ as a convex combination of $a$ and $c$ and then apply the definition of convexity.)

(b) Draw a picture illustrating this result. Interpret the result in terms of the slopes of chords of the function $f$.