STOR 655 Homework 6

1. Show that $f : \mathbb{R}^d \to \mathbb{R}$ is upper semicontinuous if and only if the super-level sets $\{x : f(x) \geq \alpha\}$ are closed for every $\alpha \in \mathbb{R}$.

2. Show that if $f_1, f_2, \ldots : \mathbb{R}^d \to \mathbb{R}$ are u.s.c. then so is $g(x) = \inf_n f_n(x)$.

3. Let X be a random variable with a finite variance and let $Y = \min(X, c)$ for some constant c. Show that the variance of Y exists and is less that or equal the variance of X. [Hint: By considering $Y - c$, show that the assertion is valid for every c if it is valid for $c = 0$. For the case $c = 0$, express X in terms of Y and $Z = \max(X, 0)$, and then consider the covariance of Y and Z.]

4. Let $\text{Bin}(n, p)$ denote the binomial distribution with parameters $n \geq 1$ and $p \in [0, 1]$. Show that for each $1 \leq k \leq n$ and each $p \in [0, 1]$ that the following identity holds:

$$P(\text{Bin}(n, p) \geq k) \;=\; \frac{n!}{(k-1)!(n-k)!} \int_0^p u^{k-1}(1-u)^{n-k} du$$

Hint: Fix $1 \leq k \leq n$. Let $f(p)$ and $g(p)$ be, respectively, the left- and right-hand sides of the equation. Show that f, g are equal when $p = 0$. Then show that $f'(p) = g'(p)$ for each p. To do this, write $f(p)$ as a sum, differentiate each summand, and then note that terms in successive summands cancel.

5. Recall that if $f : \mathcal{X} \to \mathbb{R}$ is a real-valued function then the argmax of f is the set of points in x at which f is maximized, $\arg\max_x f(x) = \{x : f(x) = \sup_{x' \in A} f(x')\}$. The argmin of f is similarly defined.

(a) Let $f : A \to \mathbb{R}$ be defined on a set $A \subseteq \mathbb{R}$ by $f(x) = x^2$. Identify the value of

$$\sup_{x \in A} f(x) \quad \text{and} \quad \arg\max_{x \in A} f(x)$$

in each of the following cases: $A = [-2, 2]$, $A = (-2, 2]$, $A = (-2, 2)$, and $A = (-3, 2]$.

(b) Let A be a bounded subset of $\mathbb{R}^d$. Identify the values of $\inf_x f(x), \sup_x f(x), \arg\min_x f(x)$, and $\arg\max_x f(x)$ for the following function:

$$f(x) = \inf_{y \in A} ||x - y||.$$

6. This problem shows how you can obtain inequalities for $\log(1+x)$ and $\log(1-x)$ from Taylor's theorem.

a. Expand the function $h(v) = \log v$ in a third order Taylor series around the point $v = 1$. (Thus you will be expressing $h(1+x)$ in terms of x, $h(1)$, $h^{'}(1)$, $h^{''}(1)$, and $h^{'''}(u)$ for some u between 1 and $1+x$. Note that x may be negative.)

b. By examining the final term in the series, use part (a) to show that $\log(1+x) \geq x - x^2/2$ for $x \geq 0$.

c. By examining the final term in the series, use part (a) to show that $\log(1-x) \leq -x - x^2/2$ for $0 \leq x < 1$.