STOR 655 Homework 3

1. Let $F_1, F_2, \ldots, F$ be one dimensional CDFs. Show that if $F(x)$ is continuous, and $F_n(x) \to F(x)$ as n tends to infinity for every $x \in \mathbb{R}$, then $\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \to 0$ as n tends to infinity. What are the implications of this fact for the central limit theorem?

2. Let X be a real-valued random variable with CDF $F(x)$. For $0 < p < 1$ define the quantile function

$$\varphi(p) \;=\; \inf\{x : F(x) \geq p\}$$

(a) Use the right-continuity of F to show that $\varphi(p) \leq x$ if and only if $p \leq F(x)$.

A number $M = M(X)$ is said to be a median of X if $P(X > M) \leq 1/2$ and $P(X < M) \leq 1/2$. Note that X may have more than one median.

(b) Show that $M = M(X)$ always exists and that $M(X)$ is unique if F is monotone increasing.

(c) Use Chebyshev's inequality to show that

$$|M(X) - \mathbb{E}X| \;\leq\; \sqrt{2}\,\mathrm{SD}(X)$$

3. Explain and establish the following relations (you may assume that all quantities are random variables rather than random vectors):

(a) $o_p(1) + o_p(1) = o_p(1)$

(b) $(1 + o_p(1))^{-1} = O_p(1)$

(c) $o_p(O_p(1)) = o_p(1)$

4. Show directly (without appealing to results about weak convergence) that if $X_1, X_2, \ldots, X \in \mathbb{R}^d$ are random vectors such that $X_n \to X$ in probability then $X_n = O_p(1)$.

5. (MKB) Let U and V be independent $\mathcal{N}(0,1)$ random variables. Define $Y = V$ and let

$$X \;=\; \begin{cases} U & \text{if } UV \geq 0 \\ -U & \text{if } UV < 0 \end{cases}$$

(a) Show that X and Y each have a standard normal distribution, but that (X, Y) is not bivariate normal.

(b) Show that X^2 and Y^2 are independent.

7. For each $k = 1, \ldots, K$ let $\{a_k(n) : n \geq 1\}$ be a sequence of real numbers. Find an inequality or equality relating

$$\limsup_{n\to\infty} \max_{1\leq k\leq K} a_k(n) \quad \text{and} \quad \max_{1\leq k\leq K} \limsup_{n\to\infty} a_k(n)$$