STOR 655 Homework 2

1. Let $X \sim \mathcal{N}_d(\mu, \Sigma)$, and let $A \in \mathbb{R}^{k \times d}$ and $B \in \mathbb{R}^{l \times d}$ be matrices. Show that the random vectors $Y = AX$ and $Z = BX$ are independent if and only if $A\Sigma B^T = 0$.

2. Show that if $X \sim \mathcal{N}_d(\mu, \Sigma)$ and $U = X^T A X$ then $\mathbb{E}U = \mathrm{tr}(A\Sigma) + \mu^T A \mu$. (It may be helpful to use the fact that $\mathrm{tr}(UV) = \mathrm{tr}(VU)$.)

3. Let $X \sim \mathcal{N}(0, 1)$ and let f be a continuously differentiable real-valued function such that $\mathbb{E}|f'(X)| < \infty$. In class we established the identity $\mathbb{E}[Xf(X)] = \mathbb{E}f'(X)$.

a. Extend the identity above to the case $X \sim \mathcal{N}(0, \sigma^2)$

b. Show that if $X \sim \mathcal{N}(\mu, \sigma^2)$ then $\mathbb{E}[(X - \mu)f(X)] = \sigma^2 \, \mathbb{E}f'(X)$

4. (Stein's Identity for Covariance) Let $X, Y \in \mathbb{R}$ be jointly normal random variables with mean zero, and let f be a continuously differentiable real-valued function satisfying appropriate integrability conditions.

a. Argue that we can write $X = aZ_1 + bZ_2$ and $Y = bZ_1 + cZ_2$ where Z_1, Z_2 are independent standard normal random variables, and a, b, c are real constants.

b. Find $\mathrm{Cov}(X, Y)$ in terms of a, b, c.

c. Show that $\mathrm{Cov}(f(X), Y) = \mathbb{E}f'(X) \, \mathrm{Cov}(X, Y)$. Hint: Use the representations of X and Y in terms of Z_1 and Z_2. Apply Stein's identity after appropriate conditioning.

d. Give some thought to what integrability conditions are needed for the covariance identity in part c.

5. (Bivariate normal distribution). Let $X = (X_1, X_2) \sim \mathcal{N}_2$ with

$$\mathbb{E}X_1 = \mu_1, \ \mathbb{E}X_1 = \mu_2, \ \mathrm{Var}(X_1) = \sigma_1^2, \ \mathrm{Var}(X_1) = \sigma_2^2, \ \mathrm{Corr}(X_1, X_2) = \rho \in [-1, 1]$$

(a) Find $\mu = \mathbb{E}X$ and $\Sigma = \mathrm{Var}(X)$ in terms of the quantities above.

(b) Find the determinant of Σ and conclude that Σ is invertible if and only if $\rho \in (-1, 1)$.

(c) Find Σ^{-1} when $\rho \in (-1, 1)$.

(d) Write down the density $f(x)$ of X in the case $\rho \in (-1, 1)$. Feel free to look up the general form of the density in a text-book, or online, and then plug in the values of μ and Σ^{-1} that you found above.

6. Let X, Y be non-negative random variables defined on the same probability space.

(a) Show that $\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)\, dt$. Hint: Use the identity $x = \int_0^\infty \mathbb{I}(x > t)\, dt$ in the integral for $\mathbb{E}X$.

(b) Let $g : [0, \infty) \to \mathbb{R}$ be a function with $g(0) = 0$ having a continuous, non-negative derivative $g'(x)$. Argue that $g(x)$ is non-negative and use the proof from part (a) to show that $\mathbb{E}g(X) = \int_0^\infty \mathbb{P}(X > t)\, g'(t)\, dt$

(c) (*Optional.*) Show that $\mathrm{Cov}(g(X), g(Y)) = \int_0^\infty \int_0^\infty H(s,t)\, g'(s)\, g'(t)\, ds\, dt$ where

$$H(s,t) = \mathbb{P}(X > s, Y > t) - \mathbb{P}(X > s)\, \mathbb{P}(Y > t)$$

7. Let U, V, W be random variables. Carefully establish the following inequalities.

(a) $\mathbb{P}(|U + V| > a + b) \;\leq\; \mathbb{P}(|U| > a) \,+\, \mathbb{P}(|V| > b)$ for every $a, b \geq 0$.

(b) $\mathbb{P}(|U\, V| > a) \;\leq\; \mathbb{P}(|U| > a/b) \,+\, \mathbb{P}(|V| > b)$ for every $a, b > 0$.

8. Let $X_1, X_2, \ldots, X$ and $Y_1, Y_2, \ldots, Y$ be d-dimensional random vectors defined on the same probability space such that $X_n \to X$ in probability and $Y_n \to Y$ in probability. Show that $(X_n + Y_n) \to (X + Y)$ in probability.

9. Let $A \subset \mathbb{R}^d$ be non-empty. Define the function $f : \mathbb{R}^d \to [0, \infty)$, representing the minimum distance from x to the set A, by

$$f(x) := \inf_{y \in A} ||x - y||$$

Show that $f(x)$ is Lipschitz with constant 1, that is, $|f(x) - f(y)| \leq ||x - y||$ for every $x, y \in \mathbb{R}^d$.