Paths, Circuits, and Connected Graphs

Paths and Circuits

Definition: Let $G = (V, E)$ be an undirected graph, vertices $u, v \in V$

- A *path* of length n from u to v is a sequence of edges

  $$e_i = \{u_{i-1}, u_i\} \in E \text{ for } i = 1, \ldots, n$$

  where initial vertex $u_0 = u$ and final vertex $u_n = v$.

- For simple graph G, represent path via vertex sequence $u_0, u_1, \ldots, u_n$.

- A path is a *circuit* if $u = v$.

- A path is *simple* if no edge e_i appears more than once (a vertex can appear more than once)

Erdős Number

Example: Collaboration graph with

- $V =$ all mathematicians
- $E =$ pairs of coauthors

Definition: The Erdős number of a mathematician u, denoted Erdős(u), is the length of the shortest path from him/her to mathematician Paul Erdős.

- Erdős$(u) = 0$ iff u is Erdős
- Erdős$(u) = 1$ iff u has written a paper with Erdős
- Erdős$(u) = 2$ iff u has *not* written a paper with Erdős, but has written a paper with a co-author of Erdős.

As of 2006 number of mathematicians with Erdős number 1, 2, and 4 was 504, 6,593, and 83K.

Connected Graphs

Definition: A graph $G = (V, E)$ is *connected* if there is a path between every two distinct vertices in V.

Connectedness important in

- Computer networks (access and security)
- Transportation networks (can't get there from here)
- Social networks (disease transmission, gossip)

Connected Graphs and Simple Paths

Theorem: If $G = (V, E)$ is undirected and connected, then there exists a simple path between every pair of vertices in V.

Proof: Fix $u, v \in V$ with $u \neq v$. Let

$$P \;=\; \{\text{all paths } p \text{ between } u \text{ and } v \text{ in } G\}$$

By assumption $P \neq \emptyset$, as G connected. Let

$$p \;=\; u_0, u_1, \ldots, u_n \text{ with } u_0 = u,\; u_n = v$$

be the vertex sequence of a path $p \in P$ with *smallest length* n.

Claim: path p is simple.

Connected Components

Connected Components

Definition: A *connected component* of a graph $G = (V, E)$ is a maximal connected subgraph, i.e., a graph H such that

- $H \leq G$
- H is connected
- No edge in G connects $V(H)$ and $\overline{V(H)}$

Note: The last condition equivalent to

- If $H \leq H' \leq G$ and H' is not equal to H, then H' is not connected.

Connectivity, cont.

Basic Facts: Let $G = (V, E)$ be an undirected graph

- If G is connected, then it has one connected component (itself)
- G can be expressed as a disjoint union of its connected components
- There is a path between vertices $u, v \in V$ if and only if they belong to the same connected component of G

Definition: Let $G = (V, E)$ be an undirected graph

- $v \in V$ is a *cut vertex* if removing v and all edges incident on it increases the number of connected components in G
- $e \in E$ is a *cut edge* if removing it increases the number of connected components in G

Paths and Isomorphisms

Given: Isomorphic graphs G_1, G_2 with isomorphism f. Note that path

$$p = u_0, u_1, \ldots, u_n$$

in graph G_1 corresponds to path

$$\tilde{p} = f(u_0), f(u_1), \ldots, f(u_n)$$

in graph G_2 and conversely. Moreover, definition of f ensures

- p simple $\Leftrightarrow$ $\tilde{p}$ simple
- p circuit $\Leftrightarrow$ $\tilde{p}$ circuit

Upshot: For $k \geq 3$ property $P_k(G) =$ G has a simple circuit of length k is a graph invariant. Useful tool to determine when two graphs are not isomorphic

Counting Paths with the Adjacency Matrix

Theorem: Let G be an undirected graph with vertices $v_1, \ldots, v_n$ and adj. matrix A. Then $\#$ paths of length r from v_i to $v_j = (i, j)$ entry of A^r.

Proof: Induction on r.

Basis step: Let $r = 1$. Then $A^r = A$ and $a_{ij} = \#$ edges between v_i and v_j, which is the number of length 1 paths from v_i to v_j

Induction step: Assume result is true for some $r \geq 1$. Note that $A^{r+1} = BA$ where $B = A^r$. Thus

$$(A^{r+1})_{ij} \; = \; (BA)_{ij} \; = \; \sum_{k=1}^{n} b_{ik}\, a_{kj}$$

Majority and Friendship Paradoxes

Majority Paradox

Example: Small town is considering a bond initiative in an upcoming election. Some residents are in favor, some are against.

Consider a poll asking the following two questions:

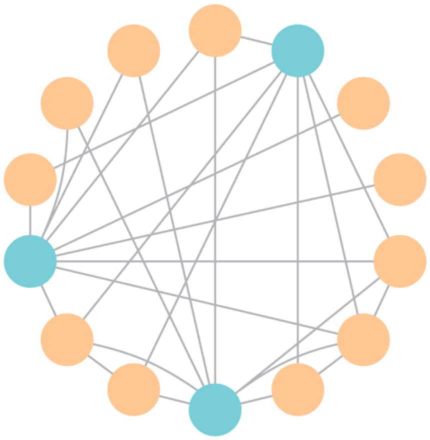
- Are you for or against the initiative?
- Do you think the initiative will pass?

Note that

- Answer to the first question depends on the opinion of the resident only.
- Answer to the second question depends on the opinion of the residents friends (and social/mainstream media they follow)

Upshot: Answers to the second question may be misleading/inaccurate

Social Network (from Wash. Post): Blue = For, Orange = Against



In this case, most residents are opposed to the initiative, but the majority of residents think the initiative will pass.

Friendship Paradox

Informal: On average, individuals in a social network have fewer friends than their friends do.

Formal: Let $G = (V, E)$ be an undirected graph. Define

- $A_1(G) =$ average degree of vertices in G
- $A_2(G) =$ average degree of the neighbors of vertices in G

Then $A_1(G) \leq A_2(G)$.

If G is a social network, then

- $A_1(G) =$ average number of friends of individuals in network
- $A_2(G) =$ average number of friends of friends in network

Proof of Friendship Paradox

Recall: $N(u)$ = neighbors of u in G, and $d(u) = |N(u)|$ = degree of u in G.

Let $|V| = n$. Then we can write

$$A_1(G) = \frac{1}{n} \sum_{u \in V} d(u) \quad \text{and} \quad A_2(G) = \frac{\sum_{u \in V} \left(\sum_{v \in N(u)} d(v) \right)}{\sum_{u \in V} |N(u)|}$$

Fact: The average $A_2(G) = \sum_{u \in V} d^2(u) \,/\, \sum_{u \in V} d(u)$

Rearranging, we find that

$$A_1(G) \leq A_2(G) \longleftrightarrow \left(\frac{1}{n} \sum_{u \in V} d(u) \right)^2 \leq \frac{1}{n} \sum_{u \in V} d^2(u)$$

Basic Lemma

To complete the proof, we appeal to a simple inequality.

Lemma: If $a_1, \ldots, a_n$ are real numbers then

$$\left(\frac{1}{n} \sum_{i=1}^{n} a_i \right)^2 \leq \frac{1}{n} \sum_{i=1}^{n} a_i^2$$

Proof: Homework.

In statistical terms this is a special case of the inequality $(EX)^2 \leq EX^2$ for a random variable X

Euler Paths

Euler Paths and Circuits

Definition: Let $G = (V, E)$ be a simple graph.

- An *Euler* path in G is a simple path that contains every edge in E.
- An *Euler* circuit in G is a simple circuit that contains every edge in E.

Example: Let G represent the map of a small town

- vertices $=$ intersections
- edges $=$ streets

Can postal worker deliver mail to residents of the town without walking any street twice?

Euler Paths and Circuits, cont.

Goal: Necessary and sufficient conditions for

- Euler paths in G
- Euler circuits in G

Punch line: There are simple conditions involving only the degree of the vertices in G

Euler Circuits and Even Degree

Theorem: Let $G = (V, E)$ be connected with $|V| \geq 2$. Then G has an Euler circuit iff every vertex has even degree.

Proof sketch ($\Leftarrow$): Fix $u_0 \in V$ and let $k = 0$. Proceed as follows.

(A) If no edge $e \in E$ contains u_k then *stop*. Otherwise

- Select an edge $e_k = \{u_k, v\} \in E$
- Let $u_{k+1} = v$ be the other endpoint of e_k
- Remove e_k from E
- Increase k by one and return to (A)

Let $e_r = \{u_r, u_{r+1}\}$ be the last edge selected. Define the path

$$p = u_0, u_1, \ldots, u_{r+1}$$

Fact 2, cont.

Claim: Path p is a circuit from u_0 to u_0, i.e., $u_{r+1} = u_0$.

Note: If p contains every edge in E, it is the desired Euler circuit. Otherwise,

- Find a circuit $\tilde{p}$ in the subgraph G_1 of G generated by the edges $E_1 = E \setminus \{\text{edges in } p\}$.
- Splice p and $\tilde{p}$ together into a single circuit.
- Continue until all edges in E are used.

Euler Paths

Theorem: A connected graph $G = (V, E)$ has an Euler path, but not an Euler circuit, iff G has exactly two vertices of odd degree.