Special Graphs

Special Graphs

Complete graph on n vertices. For $n \geq 1$, $K_n = (V, E)$ defined by

- $V = \{v_1, \ldots, v_n\}$
- $E = \{\{v_i, v_j\} : 1 \leq i < j \leq n\}$

Note $|E| = \binom{n}{2}$ maximal number of possible edges for a simple graph

Cycle of length n. For $n \geq 1$, $C_n = (V, E)$ defined by

- $V = \{v_1, \ldots, v_n\}$
- $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$

Special Graphs

The n-star. For $n \geq 1$, $S_n = (V, E)$ defined by

- $V = \{v_0, v_1, \ldots, v_n\}$
- $E = \{\{v_0, v_j\} : 1 \leq j \leq n\}$

The n-dimensional cube. For $n \geq 1$, $Q_n = (V, E)$ defined by

- $V = \{0, 1\}^n$
- $E = \{\{b_1, b_2\} : b_1, b_2$ differ in only one position$\}$

Bipartite Graphs

Definition: A simple graph $G = (V, E)$ is *bipartite* if there exist V_1, V_2 s.t.

- $V = V_1 \cup V_2$
- $V_1 \cap V_2 = \emptyset$
- If $e = \{u, v\} \in E$ then $u \in V_1$ and $v \in V_2$ or vice versa.

Terminology: V_1, V_2 is a *bipartition* of V

Idea: All edges in G are between V_1 and V_2. There are no edges between vertices in V_1 or between vertices in V_2.

Examples

Example: C_4 is bipartite, but $C_3 = K_3$ is not bipartite.

Fact: Graph G is bipartite if one can assign each vertex $v \in V$ to one of two colors such that no edge connects two vertices of the same color.

Definition: $K_{m,n}$ = complete bipartite graph with vertex set partition $V = V_1 \cup V_2$ with $|V_1| = m$ and $|V_2| = n$.

Matching and Bipartite Graphs

Example 1: Dance with n girls and n boys. Each girl knows some subset of the boys, and vice versa. (Assume that G knows B iff B knows G.)

Summary provided by a bipartite graph $G = (V_1 \cup V_2, E)$ with

- $V_1 =$ set of girls, $V_2 =$ set of boys
- $\{u, v\} \in E$ if and only if girl u knows boy v

Q: Can we group dancers into non-overlapping (boy, girl) pairs so that everyone dances with someone they know?

Matching and Bipartite Graphs

Example 2: Government needs to assign m agents to n overseas posts. Each agent lists acceptable posts.

Summary provided by a bipartite graph $G = (V_1 \cup V_2, E)$ with

- $V_1 =$ set of agents, $V_2 =$ set of posts
- $\{u, v\} \in E$ if and only if agent u finds post v acceptable

Q: Can we assign agents to posts so that every agent assigned to a different post that is acceptable to them? Note, requires $n \geq m$

Complete Matchings

Definition: Bipartite graph $G = (V_1 \cup V_2, E)$ has a *complete matching* from V_1 to V_2 if there exists a set of edges $M \subseteq E$ such that

- Each $u \in V_1$ is the endpoint of some $e \in M$
- No two edges in M share a vertex

Fact 1: Existence of a complete matching

- Implies $|V_1| = |M|$
- Requires $|V_2| \geq |V_1|$

Fact 2: If $|V_1| = |V_2|$ then there exists a complete matching M from V_1 to V_2 iff there exists a complete matching from V_2 to V_1. Then M is called “perfect”

Hall's Marriage Theorem

Goal: Find necessary and sufficient conditions for a complete matching from V_1 to V_2 in a bipartite graph $G = (V_1 \cup V_2, E)$

Definition: For every $A \subseteq V_1$ let

$$N(A) \; = \; \{v \in V_2 : \{u, v\} \in E \text{ for some } u \in A\}$$

be the set of neighbors in V_2 of the vertices $u \in A$.

Note: Complete matching requires $|A| \leq |N(A)|$ for each $A \subseteq V_1$ (*). Why?

- If $|A| < |N(A)|$ we cannot match every $u \in A$ with some $v \in N(A)$

Surprising fact: The condition (*) is also sufficient!

Hall's Marriage Theorem

Theorem: A bipartite graph $G = (V_1 \cup V_2, E)$ has a complete matching from V_1 to V_2 if and only if $|A| \leq |N(A)|$ for every $A \subseteq V_1$.

Proof: Given in the book. Fast look...

$\Rightarrow$ follows from argument above by contraposition

$\Leftarrow$ follows by strong induction on $|V_1|$

Dance

Example: Dance with n boys and n girls. Assume that every girl knows r boys, and that every boy knows r girls.

Fact: There is a perfect match of girls and boys.

Subgraphs

Definition: A subgraph of $G = (V, E)$ is a graph $H = (V', E')$ such that

- $V' \subseteq V$ and $E' \subseteq E$
- Every edge in E' connects two vertices in V'

Example: Given graph $G = (V, E)$

- Subgraph induced by vertex set $V' \subseteq V$ has edge set

$$E' = \{\{u, v\} \in E \ \text{ s.t. } u, v \in V'\}$$

- Subgraph induced by edge set $E' \subseteq E$ has vertex set

$$V' = \{\text{endpoints of edges } e \in E'\}$$

Unions of Graphs

Definition: The union of simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $H = (V, E)$ with

- $V = V_1 \cup V_2$
- $E = E_1 \cup E_2$

Notation: $H = G_1 \cup G_2$

Representing Graphs

Graph Representation

Issue: Mathematical representation of a graph $G = (V, E)$ for purposes of

- Statistical or mathematical analysis
- Storage and/or transmission

Given: Graph $G = (V, E)$ with no multiple edges

A. **Basic list:** List vertices V and edges E

B. **Adjacency list:** For each (initial) vertex, list (terminal) vertices to which it is connected.

Matrices

Recall: An $m \times n$ matrix is a rectangular array with m rows and n columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Here a_{ij} = entry in the ith row and jth column of A.

Notation: Write A in the form

$$A = [a_{ij} : 1 \leq i \leq m, 1 \leq i \leq m]$$

Adjacency Matrix

Given: Simple graph $G = (V, E)$ with vertices $V = \{v_1, v_2, \dots, v_n\}$.

Definition: The adjacency matrix A_G of G is an $n \times n$ binary matrix

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

That is, $a_{ij} = 1$ if i, j are adjacent, and $a_{ij} = 0$ otherwise

Note

- G undirected $\Rightarrow a_{ij} = a_{ji} \Rightarrow A_G$ is symmetric
- G simple $\Rightarrow a_{11} = \cdots = a_{nn} = 0$

Example

Example 1: Given a graph, find its adjacency matrix.

Example 2: Let G be a graph with vertices $\{1, 2, 3, 4\}$ and adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Draw a picture of G

Adjacency Matrices in General

Adjacency matrices can be used to represent

- Self-loops: $a_{ii} = 1$ if there is a self-loop from v_i to itself
- Multiple edges: $a_{ij} =$ number of edges between v_i and v_j
- Directed graphs: $a_{ij} = 1$ if there is a directed edge from v_i to v_j

Graph Isomorphism

Isomorphism

Given: Simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$

Question: Are G_1 and G_2 essentially the same, up to reordering of their vertices and/or differences in how they are drawn?

Definition: Graphs G_1 and G_2 are isomorphic, written $G_1 \cong G_2$, if there is a function $f : V_1 \to V_2$ such that

- f is a bijection (1:1 and onto)
- $\{u, v\} \in E_1$ if and only if $\{f(u), f(v)\} \in E_2$

Second condition says that vertices u, v are adjacent in G_1 iff the corresponding vertices $f(u), f(v)$ are adjacent in G_2

Isomorphism

Basic Properties

- Every graph is isomorphic to itself
- G_1 is isomorphic to G_2 iff G_2 is isomorphic to G_1
- If $G_1 \cong G_2$ and $G_2 \cong G_3$ then $G_1 \cong G_3$

Upshot: Isomorphism is an equivalence relation on finite graphs

More on Isomorphism

Question: Given two graphs G_1 and G_2, are they isomorphic? One can

- Exhibit an isomorphism, or
- Show that no isomorphism exists

Fact: Suppose that $G_1 = (V_1, E_1) \cong G_2 = (V_2, E_2)$. Then

- $|V_1| = |V_2|$
- $|E_1| = |E_2|$
- $(\text{deg}(v) : v \in V_1) = (\text{deg}(v) : v \in V_2)$

Graph Invariants

Definition: A property $P()$ of graphs is called a *graph invariant* if it is preserve by isomorphism, i.e.,

$$P(G_1) \text{ and } G_1 \cong G_2 \text{ imply } P(G_2)$$

Example: By previous Fact, $P(G) =$ number of vertices in G, number of edges in G, degree sequence of G, are all graph invariants.

- Idea: If $P()$ is a graph invariant and $P(G_1) \neq P(G_2)$ then G_1, G_2 are *not* isomorphic
- But agreement of invarants (e.g., number of nodes, edged, degree sequence) does *not* imply isomorphism.

Isomorphism and Adjacency

Fact: Graphs G_1 and G_2 are isomorphic if and only if one can order the vertices of G_2 so that G_1 and G_2 have the same adjacency matrix.

Connectivity and Connected Components

Paths and Circuits

Definition: Let $G = (V, E)$ be an undirected graph, vertices $u, v \in V$

- A *path* of length n from u to v is a sequence of edges

  $$e_i = \{u_{i-1}, u_i\} \in E \text{ for } i = 1, \ldots, n$$

  where initial vertex $u_0 = u$ and final vertex $u_n = v$.

- For simple graph G, represent path via vertex sequence $u_0, u_1, \ldots, u_n$.

- A path is a *circuit* if $u = v$.

- A path is *simple* if no edge e_i appears more than once (a vertex can appear more than once)

Erdős Number

Example: Collaboration graph with

- $V =$ all mathematicians
- $E =$ pairs of coauthors

Definition: The Erdős number of a mathematician u, denoted Erdős(u), is the length of the shortest path from him/her to mathematician Paul Erdős.

- Erdős$(u) = 0$ iff u is Erdős
- Erdős$(u) = 1$ iff u has written a paper with Erdős
- Erdős$(u) = 2$ iff u has *not* written a paper with Erdős, but has written a paper with a co-author of Erdős.

As of 2006 number of mathematicians with Erdős number 1, 2, and 4 was 504, 6,593, and 83K.

Connected Graphs

Definition: A graph $G = (V, E)$ is *connected* if there is a path between every two distinct vertices in V.

Connectedness important in

- Computer networks (access and security)
- Transportation networks (can't get there from here)
- Social networks (disease transmission, gossip)

Connected Graphs and Simple Paths

Theorem: If $G = (V, E)$ is undirected and connected, then there exists a simple path between every pair of vertices in V.

Proof: Fix $u, v \in V$ with $u \neq v$. Let

$$P \;=\; \{\text{all paths } p \text{ between } u \text{ and } v \text{ in } G\}$$

By assumption $P \neq \emptyset$, as G connected. Let

$$p \;=\; u_0, u_1, \ldots, u_n \text{ with } u_0 = u,\; u_n = v$$

be the vertex sequence of a path $p \in P$ with *smallest length* n.

Claim: path p is simple.

Connected Components

Definition: A *connected component* of a graph $G = (V, E)$ is a maximal connected subgraph, i.e., a graph H such that

- $H \leq G$
- H is connected
- No edge in G connects $V(H)$ and $\overline{V(H)}$

Note: The last condition equivalent to

- If $H \leq H' \leq G$ and H' is not equal to H, then H' is not connected.

Connectivity, cont.

Basic Facts: Let $G = (V, E)$ be an undirected graph

- If G is connected, then it has one connected component (itself)
- G can be expressed as a disjoint union of its connected components
- There is a path between vertices $u, v \in V$ if and only if they belong to the same connected component of G

Definition: Let $G = (V, E)$ be an undirected graph

- $v \in V$ is a *cut vertex* if removing v and all edges incident on it increases the number of connected components in G
- $e \in E$ is a *cut edge* if removing it increases the number of connected components in G

Paths and Isomorphisms

Given: Isomorphic graphs G_1, G_2 with isomorphism f. Note that path

$$p \;=\; u_0, u_1, \ldots, u_n$$

in graph G_1 corresponds to path

$$\tilde{p} \;=\; f(u_0), f(u_1), \ldots, f(u_n)$$

in graph G_2 and conversely. Moreover, definition of f ensures

- p simple $\Leftrightarrow$ $\tilde{p}$ simple
- p circuit $\Leftrightarrow$ $\tilde{p}$ circuit

Upshot: For $k \geq 3$ the property $P_k(G) =$ G has a simple circuit of length k is a graph invariant. A useful tool to determine when two graphs are not isomorphic

Counting Paths with the Adjacency Matrix

Theorem: Let G be an undirected graph with vertices $v_1, \ldots, v_n$ and adj. matrix A. Then # paths of length r from v_i to $v_j = (i, j)$ entry of A^r.

Proof: Induction on r.

Basis step: Let $r = 1$. Then $A^r = A$ and $a_{ij} = \#$ edges between v_i and v_j, which is the number of length 1 paths from v_i to v_j

Induction step: Assume result is true for some $r \geq 1$. Note that $A^{r+1} = BA$ where $B = A^r$.