Graphs and Networks

Graphs

A graph is a set of vertices, select pairs of which are connected by edges.

Formal: A graph is a pair $G = (V, E)$ where

- V is a non-empty set of *vertices*
- E is a set of *edges* connecting pairs of vertices

Two flavors of graphs

- *Undirected*: Each $e \in E$ is an unordered pair $\{u, v\}$ with $u, v \in V$
- *Directed*: Each $e \in E$ is an ordered pair (u, v) with $u, v \in V$

Idea: Graphs capture *pairwise* relationships (edges) between a set of objects (vertices)

Drawing Graphs

Drawing a graph $G = (V, E)$

- Vertices $u, v \in V$ represented as points
- Undirected edge $e = \{u, v\} \in E$ represented as a line between u to v
- Directed edge $e = (u, v) \in E$ represented as an arrow from u to v

Example: Draw the graph G with

- $V = \{1, 2, 3, 4\}$, $E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{3, 3\}\}$
- $V = \{1, 2, 3\}$, $E = \{(1, 2), (2, 1), (3, 2), (3, 3)\}$

Examples

Organizational Hierarchy

- Vertices $V =$ employees of a company
- Edges $E =$ all (u, v) such that u reports to v

Airline Network

- Vertices $V =$ cities in the US to which an airline flies
- Edges $E =$ all (u, v) such that there is a flight from u to v

Facebook Network

- Vertices $V =$ all users of Facebook
- Edges $E =$ all (u, v) such that u is friends with v

Examples

Co-authorship Network

- All statisticians
- Edges $E =$ all $\{u, v\}$ such that u and v have co-authored a paper

Disease Transmission

- Vertices $V =$ residents of the U.S.
- Edges $E =$ all $\{u, v\}$ such that u and v have been in close proximity

Internet

- Vertices $V =$ computers worldwide
- Edges $E =$ all $\{u, v\}$ such that u and v have a direct connection

Terminology

- Graph G is *finite* if V is finite and *infinite* otherwise
- Vertices u, v called *endpoints* of edge $e = \{u, v\}$ or $e = (u, v)$
- An edge $e = \{u, u\}$ or $e = (u, u)$ is called a (self) *loop*
- In some cases, a graph can have multiple edges between two nodes
- Graphs are often referred to as *networks*, vertices as *nodes*, and edges as *links*

Adjacency and Degree

Adjacency

Given: Undirected graph $G = (V, E)$

- Vertices $u, v \in V$ are *adjacent* if $\{u, v\} \in E$
- Edge $\{u, v\} \in E$ is *incident* on vertices u, v
- Vertices $u, v \in V$ are endpoints of the edge $\{u, v\} \in E$

Definition: A graph G is *simple* if it is undirected and has no self-loops or multiple edges.

Degree

Definition: The degree of a vertex u in a graph $G = (V, E)$ is the number of edges incident on u, with loops counting twice

$$\text{deg}(u) \; = \; \sum_{v \in V} \mathbb{I}(\{u, v\} \in E)$$

Terminology: u is *isolated* if $\text{deg}(u) = 0$, and is a *pendant* if $\text{deg}(u) = 1$.

Fact: If G has no loops or multiple edges then $\text{deg}(u) \leq |V| - 1$

Basic Facts

Fact 1: If $G = (V, R)$ is simple then $|E| \leq \binom{|V|}{2}$

Fact 2: If $G = (V, R)$ is simple then at least two vertices have the same degree.

Proof: Let $V = \{v_1, \dots v_n\}$ be ordered such that $d_1 \leq d_2 \leq \cdots \leq d_n$ where $d_i =$ deg(v_i). If $d_i \neq d_j$ for $i \neq j$, then

$$d_1 < d_2 < \ldots < d_n$$

Consider two cases

1. $d_1 \geq 1$
2. $d_1 = 0$

Handshaking Theorem

Fact: If $G = (V, E)$ is undirected then

$$\sum_{v \in V} \text{deg}(v) \;=\; 2\,|E|$$

Cor: If $G = (V, R)$ is undirected then the sum of the odd degrees is even

$$\sum_{v \text{ s.t. } \text{deg}(v) \text{ is odd}} \text{deg}(v) \text{ is even}$$

Example: A graph has 6 vertices with degrees $5, 4, 3, 2, 2, 0$. How many edges does it have?

Directed Graphs

Given: Directed graph $G = (V, E)$. If $e = (u, v) \in E$ we say

- u is the initial vertex of e
- v is the terminal vertex of e

Definition:

- In-degree $\deg^{-}(v) = |\{e \in E : e = (u, v)$ some $u \in V\}|$
  number of edges terminating at v
- Out-degree $\deg^{+}(v) = |\{e \in E : e = (v, w)$ some $w \in V\}|$
  number of edges originating at v

Note: Loops contribute one to in-degree and out-degree.

In-degrees and Out-degrees

Fact: In a directed graph,

$$|E| \;=\; \sum_{v \in V} \mathsf{deg}^{+}(v) \;=\; \sum_{v \in V} \mathsf{deg}^{-}(v)$$

Special Graphs

Special Graphs

Complete graph on n vertices. For $n \geq 1$, $K_n = (V, E)$ defined by

- $V = \{v_1, \ldots, v_n\}$
- $E = \{\{v_i, v_j\} : 1 \leq i < j \leq n\}$

Note $|E| = \binom{n}{2}$ maximal number of possible edges for a simple graph

Cycle of length n. For $n \geq 1$, $C_n = (V, E)$ defined by

- $V = \{v_1, \ldots, v_n\}$
- $E = \{\{v_1, v_2\}, \{v_1, v_2\}, \ldots, \{v_1, v_2\}\}$

Special Graphs

The n-star. For $n \geq 1$, $S_n = (V, E)$ defined by

- $V = \{v_0, v_1, \ldots, v_n\}$
- $E = \{\{v_0, v_j\} : 1 \leq j \leq n\}$

The n-dimensional cube. For $n \geq 1$, $Q_n = (V, E)$ defined by

- $V = \{0, 1\}^n$
- $E = \{\{b_1, b_2\} : b_1, b_2$ differ in only one position$\}$

Bipartite Graphs

Definition: A simple graph $G = (V, E)$ is *bipartite* if there exist V_1, V_2 s.t.

- $V = V_1 \cup V_2$
- $V_1 \cap V_2 = \emptyset$
- If $e = \{u, v\} \in E$ then $u \in V_1$ and $v \in V_2$ or vice versa.

Terminology: V_1, V_2 is a *bipartition* of V

Idea: All edges in G are between V_1 and V_2. There are no edges between vertices in V_1 or between vertices in V_2.

Examples

Example: C_4 is bipartite, but $C_3 = K_3$ is not bipartite.

Fact: Graph G is bipartite if one can assign each vertex $v \in V$ to one of two colors such that no edge connects two vertices of the same color.

Definition: $K_{m,n} =$ complete bipartite graph with vertex set partition $V = V_1 \cup V_2$ with $|V_1| = m$ and $|V_2| = n$.

Matching and Bipartite Graphs

Example 1: Dance with n girls and n boys. Each girl knows some subset of the boys, and vice versa. (Assume that G knows B iff B knows G.)

Summary provided by a bipartite graph $G = (V_1 \cup V_2, E)$ with

- $V_1 =$ set of girls, $V_2 =$ set of boys
- $\{u, v\} \in E$ if and only if girl u knows boy v

Q: Can we group dancers into non-overlapping (boy, girl) pairs so that everyone dances with someone they know?

Matching and Bipartite Graphs

Example 2: Government needs to assign m agents to n overseas posts. Each agent lists acceptable posts.

Summary provided by a bipartite graph $G = (V_1 \cup V_2, E)$ with

- $V_1 =$ set of agents, $V_2 =$ set of posts
- $\{u, v\} \in E$ if and only if agent u finds post v acceptable

Q: Can we assign agents to posts so that every agent assigned to a different post that is acceptable to them? Note, requires $n \geq m$

Complete Matchings

Definition: Bipartite graph $G = (V_1 \cup V_2, E)$ has a *complete matching* from V_1 to V_2 if there exists a set of edges $M \subseteq E$ such that

- Each $u \in V_1$ is the endpoint of some $e \in M$
- No two edges in M share a vertex

Fact 1: Existence of a complete matching

- Implies $|V_1| = |M|$
- Requires $|V_2| \geq |V_1|$

Fact 2: If $|V_1| = |V_2|$ then there exists a complete matching M from V_1 to V_2 iff there exists a complete matching from V_2 to V_1. M called "perfect"

Hall's Marriage Theorem

Goal: Find necessary and sufficient conditions for a complete match in a bipartite graph $G = (V_1 \cup V_2, E)$

Definition: For every $A \subseteq V_1$ let

$$N(A) \ = \ \{v \in V_2 : \{u, v\} \in E \text{ for some } u \in A\}$$

be the set of neighbors of the vertices $u \in A$ in V_2.

Note: If some set $A \subseteq V_1$ with $|A| = k$ has fewer than k neighbors in V_2 then a complete match is not possible: there is no way to match every $u \in A$ with some $v \in N(A)$.

Surprising fact: This condition is also necessary.

Hall's Marriage Theorem

Theorem: A bipartite graph $G = (V_1 \cup V_2, E)$ has a complete matching from V_1 to V_2 if and only if $|A| \le |N(A)|$ for every $A \subseteq V_1$.

Proof: Given in the book. Fast look...

$\Rightarrow$ follows from argument above by contraposition

$\Leftarrow$ follows by strong induction on $|V_1|$

Dance

Example: Suppose that there are n boys and n girls at a dance, that every girl knows r boys, and that every boy knows r girls.

Fact: There is a perfect match of girls and boys.