Permutations and Combinations

Permutations

Definition: Let S be a set with n elements

- A *permutation* of S is an ordered list (arrangement) of its elements
- For $r = 1, \ldots, n$ an r-*permutation* of S is an ordered list (arrangement) of r elements of S.

Definition: Let $P(n,r) =$ # of r-permutations of an n element set

Fact: $P(n,r) = n\,(n-1)\,\cdots(n-r+1)$

Corollary: $P(n,n) = n!$ and in general $P(n,r) = n!/(n-r)!$

Combinations

Definition: Let S be a set with n elements. For $0 \leq r \leq n$ an r-*combination* of S is a (unordered) subset of S with r elements.

Definition: Let $C(n, r) =$ # of r-combinations of an n element set

Fact: $C(n, 0) = 1$ (the empty set) and with convention $0! = 1$ we can write

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!\,(n-r)!} \quad 0 \leq r \leq n$$

Corollary: $C(n, r) = C(n, n - r)$

Coin Flipping Example

Example: Flip coin 12 times

Q1: What is the number of possible outcomes with 5 heads?

Let $S = \{1, 2, \ldots, 12\}$ be index/position of 12 flips

Outcome with 5 heads obtained as follows:

- Select 5 positions from S
- Assign H to these positions and T to all other positions

Q2: What is the number of possible outcomes with at least 3 tails?

Q3: Number of possible outcomes with an equal number of heads and tails?

Hat with Cards Example

Example: Hat contains 20 cards: 1 red, 5 blue, 14 white. Cards are removed from the hat one at a time and placed side by side, in order.

How many color sequences are there under the following restrictions

1. No restrictions
2. Red card in position 2
3. Blue cards in positions $8, 15, 16$
4. Blue cards in positions $3, 4$ and white cards in positions $8, 9, 10$

Binomial Coefficients and Identities

Binomial Theorem

Example

$$(x+y)^2 \;=\; x^2 + 2xy + y^2 \;=\; \binom{2}{0} x^2 y^0 \;+\; \binom{2}{1} x^1 y^1 \;+\; \binom{2}{2} x^0 y^1$$

Example

$$(x+y)^3 \;=\; x^3 + 3x^2 y + 3xy^2 + y^3 \;=\; \sum_{k=0}^{3} \binom{3}{k} x^{n-k} y^k$$

Binomial Theorem: For all $x, y \in \mathbb{R}$ and $n \geq 0$

$$(x+y)^n \quad = \quad \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Corollaries of Binomial Theorem

Fact 1: $2^n = \sum_{j=0}^{n} \binom{n}{j}$

Fact 2: $3^n = \sum_{j=0}^{n} \binom{n}{j} 2^j$

Fact 3: $\sum_{j=0}^{n} (-1)^j \binom{n}{j} = 0$

Corollary: Sum of $\binom{n}{j}$ over even j is equal to sum of $\binom{n}{j}$ over odd j

Monotonicity of Binomial Coefficients

Fact: Let $n \geq 1$. Then

$$\binom{n}{r} \leq \binom{n}{r+1} \quad \text{if and only if} \quad r \leq (n-1)/2$$

Idea: Binomial coefficients increase as r goes from 0 to $(n-1)/2$, then decrease as r goes from $(n-1)/2$ to n.

Pascal's Triangle

Pyramid with nth row equal to the binomial coefficients $\binom{n}{0}, \ldots, \binom{n}{n}$

$$\binom{0}{0}$$
$$\binom{1}{0}\binom{1}{1}$$
$$\binom{2}{0}\binom{2}{1}\binom{2}{2}$$
$$\binom{3}{0}\binom{3}{1}\binom{3}{2}\binom{3}{3}$$
$$\ldots$$

Pascal's Identity: If $1 \leq k \leq n$ then $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

The identity says that every entry of Pascal's triangle can be obtained by adding the two entries above it.

Using Pascal's Identity

Example: Show that

$$\binom{2n+2}{n+1} = 2\binom{2n}{n} + 2\binom{2n}{n+1}$$

Vandermonde Identity

Vandermonde Identity: If $1 \le r \le m, n$ then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}$$

Proof: Let $S = \{0,1\}^{m+n}$. For $k = 0, \ldots, r$ define

- $A = \{b \in S$ with r ones$\}$
- $A_k = \{b \in S$ with $(r-k)$ ones in first m bits, k ones in last n bits$\}$

Note that

- $A = A_0 \cup A_1 \cup \cdots \cup A_r$
- $A_i \cap A_j = \emptyset$ if $i \neq j$

More Identities

Corollary: For $n \geq 1$ we have $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$

Fact: If $1 \leq r \leq n$ then $\binom{n+1}{r+1} = \sum_{k=r}^{n} \binom{k}{r}$

Proof: Let $S = \{0,1\}^{n+1}$. For $k = r, \ldots, n$ define

- $A_k = \{b \in S$ having $(r+1)$ ones, with last one in positions $k+1\}$
- $A = \{b \in S$ with $(r+1)$ ones$\}$

Note that

- $A_i \cap A_j = \emptyset$ if $i \neq j$
- $A = A_r \cup A_1 \cup \cdots \cup A_n$

More Permutations and Combinations

Permutations of Indistinguishable Objects

Example: Number of distinct rearrangements of the letters in SUPPRESS?

Fact: Suppose we are given n objects of k different types where

- there are n_j objects of type j
- objects of the same type are indistinguishable

Then the number of distinct permutations of these objects is given by the *multinomial coefficient*

$$\binom{n}{n_1 \cdots n_k} := \frac{n!}{n_1! \cdots n_k!}$$

(Numerator $=$ # permutations of n distinct objects. Denominator accounts for the fact that, once their positions are fixed, all $n_r!$ permutations of type r objects yield the same pattern.)

The Multinomial Theorem

Theorem: If $n \geq 1$ and $x_1, \ldots, x_m \in \mathbb{R}$ then

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{n_1 + \cdots + n_m = n} \binom{n}{n_1 \cdots n_m} x_1^{n_1} \cdots x_m^{n_m}$$

Note: The Binomial Theorem is the special case where $m = 2$.

Objects in Boxes

General Question: How many ways are there to distribute n objects into k boxes?

Four cases: Objects and boxes can be

- distinguishable (labeled)
- indistinguishable (unlabeled)

Case 1: Objects and Boxes are Distinguishable

Q: How many ways are there to distribute $n = n_1 + \cdots n_k$ distinguishable objects into k distinct boxes so that box r has n_r objects (order of objects in a box is not important)?

A: Multinomial coefficient

$$\binom{n}{n_1 \cdots n_k}$$

Example: Given standard deck of 52 cards, how many ways are there to deal hands of 5 cards to 3 different players?

Example: How many ways are there to place n distinct objects into k distinguishable boxes?

Case 2: Objects Indistinguishable, Boxes Distinguishable

Q: How many ways are there to place n indistinguishable objects into k distinguishable boxes?

A: Binomial coefficient

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

Indistinguishable Objects in Distinguishable Boxes

Proof: For $1 \leq j \leq k$ let $n_j = \#$ objects in box j.

- There is a 1:1 correspondence between assignments of n objects to k boxes and k-tuples $(n_1, \ldots, n_k)$ such that

$$n_j \geq 0 \text{ and } \sum_{j=1}^{n} n_j = n \quad (*)$$

- There is a 1:1 correspondence between k-tuples $(n_1, \ldots, n_k)$ satisfying $(*)$ and the arrangement of $k-1$ "bars" and n "stars"

$$(n_1, \ldots, n_k) \Leftrightarrow * \cdots * \,|\, * \cdots * \,|\, \cdots \,|\, * \cdots *$$

- The number of such arrangements is

$$\binom{n+k-1}{k-1}$$

Examples

Example: How many integer solutions are there to the equation $x_1 + x_2 + x_3 = 18$ under the following constraints?

1. $x_1, x_2, x_3 \geq 0$
2. $x_1, x_2 \geq 0$ and $x_3 = 4$
3. $x_1 \geq 1, x_2 \geq 2, x_3 \geq 3$

Example: A drawer contains red, blue, green, and yellow socks. Assume that socks are indistinguishable apart from color and there are at least 8 socks of each color.

How many ways can 8 socks be chosen from the drawer (order unimportant)?

Graphs and Networks

Graphs

A graph is a set of vertices, select pairs of which are connected by edges.

Formal: A graph is a pair $G = (V, E)$ where

- V is a non-empty set of *vertices*
- E is a set of *edges* connecting pairs of vertices

Two flavors of graphs

- *Undirected*: Each $e \in E$ is an unordered pair $\{u, v\}$ with $u, v \in V$
- *Directed*: Each $e \in E$ is an ordered pair (u, v) with $u, v \in V$

Idea: Graphs capture *pairwise* relationships (edges) between a set of objects (vertices)

Drawing Graphs

Drawing a graph $G = (V, E)$

- Vertices $u, v \in V$ represented as points
- Undirected edge $e = \{u, v\} \in E$ represented as a line between u to v
- Directed edge $e = (u, v) \in E$ represented as an arrow from u to v

Example: Draw the graph G with

- $V = \{1, 2, 3, 4\}$, $E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{3, 3\}\}$
- $V = \{1, 2, 3\}$, $E = \{(1, 2), (2, 1), (3, 2), (3, 3)\}$

Examples

Organizational Hierarchy

- Vertices $V =$ employees of a company
- Edges $E =$ all (u, v) such that u reports to v

Airline Network

- Vertices $V =$ cities in the US to which an airline flies
- Edges $E =$ all (u, v) such that there is a flight from u to v

Facebook Network

- Vertices $V =$ all users of Facebook
- Edges $E =$ all (u, v) such that u is friends with v

Examples

Co-authorship Network

- All statisticians
- Edges $E =$ all $\{u, v\}$ such that u and v have co-authored a paper

Disease Transmission

- Vertices $V =$ residents of the U.S.
- Edges $E =$ all $\{u, v\}$ such that u and v have been in close proximity

Internet

- Vertices $V =$ computers worldwide
- Edges $E =$ all $\{u, v\}$ such that u and v have a direct connection

Terminology

- Graph G is *finite* if V is finite and *infinite* otherwise
- Vertices u, v called *endpoints* of edge $e = \{u, v\}$ or $e = (u, v)$
- An edge $e = \{u, u\}$ or $e = (u, u)$ is called a (self) *loop*
- In some cases, a graph can have multiple edges between two nodes
- Graphs are often referred to as *networks*, vertices as *nodes*, and edges as *links*

Adjacency and Degree

Adjacency

Given: Undirected graph $G = (V, E)$

- Vertices $u, v \in V$ are *adjacent* if $\{u, v\} \in E$
- Edge $\{u, v\} \in E$ is *incident* on vertices u, v
- Vertices $u, v \in V$ are endpoints of the edge $\{u, v\} \in E$

Definition: A graph G is *simple* if it is undirected and has no self-loops or multiple edges.

Degree

Definition: The degree of a vertex u in a graph $G = (V, E)$ is the number of edges incident on u, with loops counting twice

$$\text{deg}(u) \; = \; \sum_{v \in V} \mathbb{I}(\{u, v\} \in E)$$

Terminology: u is *isolated* if $\text{deg}(u) = 0$, and is a *pendant* if $\text{deg}(u) = 1$.

Fact: If G has no loops or multiple edges then $\text{deg}(u) \leq |V| - 1$