Introduction to Decision Sciences

Lecture 11

Andrew Nobel

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Basics of Counting

Product Rule

Product Rule: Suppose that the elements of a collection S can be specified by a sequence of k steps such that

- There are n_j possibilities at step j
- The selections made at steps $1, \ldots, j$ do not affect the *number* of possibilities at step $j+1$

Then S has $n_1 \, n_2 \cdots n_k$ elements

Example: Cartesian product of sets $A_1, \ldots, A_k$ is

$$A_1 \times \cdots \times A_k \;=\; \{(a_1, \ldots, a_k) : a_1 \in A_1, \ldots, a_k \in A_k\}$$

By product rule $|A_1 \times \cdots \times A_k| \;=\; |A_1| \cdots |A_k|$

Example: Counting Functions

Given: Finite sets $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$

1. What is the number of functions $f : A \to B$?
2. What is the number of one-to-one functions $f : A \to B$?
3. What is the number of onto functions $f : A \to B$?

Indicator Functions

Definition: The *indicator function* of a proposition q is given by

$$I(q) = \begin{cases} 1 & \text{if } q \text{ is true} \\ 0 & \text{if } q \text{ is false} \end{cases}$$

Example: Find the size of 2^S for a finite set $S = \{s_1, \ldots, s_n\}$

- Define function $f : 2^S \to \{0,1\}^n$ from 2^S to binary n-tuples by

$$f(A) \;=\; (I(s_1 \in A), I(s_2 \in A), \ldots, I(s_n \in A))$$

- Can check that $f()$ is one-to-one and onto, so

$$|2^S| \;=\; |\{0,1\}^n| \;=\; 2^n \;=\; 2^{|S|}$$

Sum Rule

Sum Rule: S'pose that each element of a collection S is one of k types, and

- There are n_j elements of type j
- No element can be of more than one type.

Then $|S| = n_1 + \cdots + n_k$

Equivalent Form: If $S = A_1 \cup \cdots \cup A_k$ where $A_i \cap A_j$ for $i \neq j$ then

$$|S| = |A_1| + \cdots + |A_k|$$

Example: How many binary sequences b of length 6 begin with 01 or 001?

Inclusion-Exclusion

Fact: If A, B are sets then $|A \cup B| = |A| + |B| - |A \cap B|$.

General Form: For sets $A_1, \ldots, A_n$

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \cdots i_k \leq n} |A_{i_1} \cap \cdots \cap A_{i_k}|$$

Example: How many sequences $b \in \{0,1\}^8$ begin with 00 or end with 11?

Example: Suppose you have 5 friends who play golf, 8 who play tennis, and 3 who play both. How many of your friends play golf or tennis?

The Pigeonhole Principle

Fact: If $k+1$ objects are placed in k boxes then one box must contain at least two objects.

Why? Let $n_j = \#$ objects in box j. If each $n_j \leq 1$ then

$$\sum_{j=1}^{k} n_j \;\leq\; \sum_{j=1}^{k} 1 \;=\; k \;<\; k+1.$$

Examples

1. Among 13 people, at least two have their birthday in the same month.
2. Among 11 people, at least two have the same last digit in their phone number.
3. If $f : A \to B$ is a function and $|B| < |A|$ then f is not one-to-one

More Elaborate Applications of PHP

Fact: At a party with $n \geq 2$ guests there are at least two people with the same number of friends. (Assume a is friends with b iff b is friends with a.)

Why? For $1 \leq j \leq n$ let $m_j = \#$ friends of guest j at the party

- *Case 1:* Everybody has at least 1 friend. Then each of $m_1, \ldots, m_n$ is between 1 and $n-1$. Two of these numbers must be the same
- *Case 2:* Somebody has no friends. Then each of $m_1, \ldots, m_n$ is between 0 and $n-2$. Two of these numbers must be the same

Triathlon Training (adapted from website of P. Talwalkar)

Gary is training for a triathlon. Over a 30 day period he trains at least once every day, and 45 times in total.

Claim: There is a set of consecutive days when Gary trains exactly 14 times.

Why? For $1 \leq j \leq 30$ let $s_j = \#$ workouts by end of day j. We know that

$$1 \leq s_1 < s_2 < \cdots < s_{29} < s_{30} = 45.$$

Adding 14 to each term gives $15 \leq s_1 + 14 < \cdots < s_{30} + 14 = 59$.

Upshot: The numbers $s_1, \ldots, s_{30}, s_1 + 14, \ldots, s_{30} + 14$ lie between 1 and 59

By PHP, two of these 60 numbers are the same, and as the numbers within each group are strictly increasing, there must be some i, j such that

$$s_i = s_j + 14 \quad \Leftrightarrow \quad s_i - s_j = 14$$

Another Application of PHP

Fact: For each $n \geq 1$ there is an $r \geq 1$ s.t. the decimal expansion of rn contains only 0s and 1s

Generalized Pigeon Hole Principle

Generalized PHP: If N objects are placed in k boxes, then there is a box containing at least $\lceil N/k \rceil$ objects

Proof: Similar to regular PHP.

Example: Among a class of 60 people at least $\lceil 60/5 \rceil = 12$ will receive one of the letter grades A, B, C, D, F.

Example: Rolling a 6-sided die

1. How many rolls guarantee that we see some number at least 4 times?
2. How many rolls guarantee that we see 1 at least 2 times?

Permutations and Combinations

Permutations

Definition: Let S be a set with n elements

- A *permutation* of S is an ordered list (arrangement) of its elements
- For $r = 1, \ldots, n$ an r-*permutation* of S is an ordered list (arrangement) of r elements of S.

Definition: Let $P(n, r) =$ # of r-permutations of an n element set

Fact: $P(n, r) = n\,(n-1)\,\cdots\,(n-r+1)$

Corollary: $P(n, n) = n!$ and in general $P(n, r) = n!/(n-r)!$

Combinations

Definition: Let S be a set with n elements. For $0 \leq r \leq n$ an r-*combination* of S is a (unordered) subset of S with r elements.

Definition: Let $C(n, r) =$ # of r-combinations of an n element set

Fact: $C(n, 0) = 1$ (the empty set) and with convention $0! = 1$ we can write

$$C(n, r) \;=\; \binom{n}{r} \;=\; \frac{n!}{r!\,(n-r)!} \quad 0 \leq r \leq n$$

Corollary: $C(n, r) = C(n, n-r)$

Coin Flipping Example

Example: Flip coin 12 times

Q1: What is the number of possible outcomes with 5 heads?

Let $S = \{1, 2, \ldots, 12\}$ be index/position of 12 flips

Outcome with 5 heads obtained as follows:

- Select 5 positions from S
- Assign H to these positions and T to all other positions

Q2: What is the number of possible outcomes with at least 3 tails?

Q3: Number of possible outcomes with an equal number of heads and tails?

Hat with Cards Example

Example: Hat contains 20 cards: 1 red, 5 blue, 14 white. Cards are removed from the hat one at a time and placed side by side, in order.

How many color sequences are there under the following restrictions

1. No restrictions
2. Red card in position 2
3. Blue cards in positions $8, 15, 16$
4. Blue cards in positions $3, 4$ and white cards in positions $8, 9, 10$

Binomial Coefficients and Identities

Binomial Theorem

Example

$$(x+y)^2 \;=\; x^2+2xy+y^2 \;=\; \binom{2}{0}x^2y^0 \;+\; \binom{2}{1}x^1y^1 \;+\; \binom{2}{2}x^0y^1$$

Example

$$(x+y)^3 \;=\; x^3+3x^2y+3xy^2+y^3 \;=\; \sum_{k=0}^{3}\binom{3}{k}x^{n-k}y^k$$

Binomial Theorem: For all $x, y \in \mathbb{R}$ and $n \geq 0$

$$(x+y)^n \quad = \quad \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$$

Corollaries of Binomial Theorem

Fact 1: $2^n = \sum_{j=0}^{n} \binom{n}{j}$

Fact 2: $3^n = \sum_{j=0}^{n} \binom{n}{j} 2^j$

Fact 3: $\sum_{j=0}^{n} (-1)^j \binom{n}{j} = 0$

Corollary: Sum of $\binom{n}{j}$ over even j is equal to sum of $\binom{n}{j}$ over odd j

Monotonicity of Binomial Coefficients

Fact: Let $n \geq 1$. Then

- $C(n,r) \leq C(n,r+1)$ if $r \leq (n-1)/2$
- $C(n,r) \geq C(n,r+1)$ if $r \geq (n-1)/2$

Idea: Binomial coefficients increase as r goes from 0 to $(n-1)/2$, then decrease as r goes from $(n-1)/2$ to n.

Pascal's Triangle

Pyramid with nth row equal to the binomial coefficients $\binom{n}{0}, \ldots, \binom{n}{n}$

$$\binom{0}{0}$$
$$\binom{1}{0}\binom{1}{1}$$
$$\binom{2}{0}\binom{2}{1}\binom{2}{2}$$
$$\binom{3}{0}\binom{3}{1}\binom{3}{2}\binom{3}{3}$$
$$\ldots$$

Pascal's Identity: If $1 \leq k \leq n$ then $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

The identity says that every entry of Pascal's triangle can be obtained by adding the two entries above it.

Using Pascal's Identity

Example: Show that

$$\frac{1}{2} \cdot \binom{2n+2}{n+1} = \binom{2n}{n} + \binom{2n}{n+1}$$

Vandermonde Identity

Vandermonde Identity: If $1 \le r \le m, n$ then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Proof: Let $S = \{0,1\}^n$. For $k = 0, \ldots, r$ define

- $A = \{b \in S$ with r ones$\}$
- $A_k = \{b \in S$ with $(r-k)$ ones in first m bits, k ones in last n bits$\}$

Note that

- $A = A_0 \cup A_1 \cup \cdots \cup A_r$
- $A_i \cap A_j = \emptyset$ if $i \neq j$

More Identities

Corollary: For $n \geq 1$ we have $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$

Fact: If $1 \leq r \leq n$ then $\binom{n+1}{r+1} = \sum_{k=r}^{n} \binom{k}{r}$

Why? Let $S = \{0,1\}^{n+1}$. For $k = r, \ldots, n$ define

- $A_k = \{b \in S$ having $(r+1)$ ones, with last one in positions $k+1\}$
- $A = \{b \in S$ with $(r+1)$ ones$\}$

Note that

- $A = A_r \cup A_1 \cup \cdots \cup A_n$
- $A_i \cap A_j = \emptyset$ if $i \neq j$

More Permutations and Combinations

Permutations of Indistinguishable Objects

Example: How many distinct rearrangements are there of the letters in the word SUPRESS?

Fact: Suppose we are given n objects of k different types where

- there are n_j objects of type j
- objects of the same type are indistinguishable

Then the number of distinct permutations of these objects is given by the *multinomial coefficient*

$$\binom{n}{n_1 \cdots n_k} := \frac{n!}{n_1! \cdots n_k!}$$

The Multinomial Theorem

Theorem: If $n \geq 1$ and $x_1, \ldots, x_m \in \mathbb{R}$ then

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{n_1 + \cdots n_m = n} \binom{n}{n_1 \cdots n_m} \cdot x_1^{n_1} \cdots x_m^{n_m}$$

Note: The Binomial Theorem is the special case where $m = 2$.

Objects in Boxes

General Question: How many ways are there to distribute n objects into k boxes?

Four cases: Objects and boxes can be

- distinguishable (labeled)
- indistinguishable (unlabeled)

Case 1: Objects and Boxes are Distinguishable

Qu: How many ways are there to distribute $n = n_1 + \cdots n_k$ distinguishable objects into k distinct boxes so that box r has n_r objects?

Ans: Multinomial coefficient

$$\binom{n}{n_1 \cdots n_k}$$

Example: Given standard deck of 52 cards, how many ways are there to deal hands of 5 cards to 3 different players?

Example: How many ways are there to place n distinct objects into k distinguishable boxes?

Case 2: Objects Indistinguishable, Boxes Distinguishable

Qu: How many ways are there to place n indistinguishable objects into k distinguishable boxes?

Ans: For $1 \leq j \leq k$ let $n_j = \#$ objects in box j.

- There is a 1:1 correspondence between assignments of objects to boxes and k-tuples $(n_1, \ldots, n_k)$ such that

$$n_j \geq 0 \text{ and } \sum_{j=1}^{n} n_j = n \quad (*)$$

- There is a 1:1 correspondence between k-tuples $(n_1, \ldots, n_k)$ satisfying $(*)$ and the arrangement of $k-1$ “bars” and n “stars”

$$(n_1, \ldots, n_k) \Leftrightarrow * \cdots * \,|\, * \cdots * \,|\, \cdots \,|\, * \cdots *$$