Introduction to Decision Sciences

Lecture 10

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Mathematical Induction

Given: Propositional function $P(n)$ with domain $\mathbb{N}_+$

Basis step: Show that $P(1)$ is true

Inductive step: Show that $P(k) \to P(k+1)$ is true for every $k \geq 1$

- assume that $P(k)$ is true "inductive hypothesis"
- establish that $P(k+1)$ is true

Conclusion: $P(n)$ is true for every $n \in \mathbb{N}_+$

We can view induction as a (new) rule of inference, namely,

$$[P(1) \land \forall k\,(P(k) \to P(k+1))] \to \forall n\, P(n)$$

Validity of Induction

Informal: Ladder/Dominos

- $P(1)$ is true by Basis step
- $P(1) \to P(2)$ is true by Inductive step, so $P(2)$ is true
- $P(2) \to P(3)$ is true by Inductive step, so $P(3)$ is true
- $P(3) \to P(4)$ is true by Inductive step, so $P(4)$ is true
- and so on...

Conclude: $P(n)$ is true for every n

Validity of Induction

Formal: Suppose that basis and inductive steps hold but $\forall n\, P(n)$ is F

- Then $S = \{n : P(n)$ is F$\}$ is non-empty
- By well-ordering, S has smallest element m
- By Basis step, $P(1)$ is true so $m \geq 2$
- Definition of S implies $P(m-1)$ is T and we know $m - 1 \geq 1$
- Inductive step then implies $P(m)$ is T, a contradiction
- Conclude that $\forall n\, P(n)$ is T

Examples

Example 1: Sum of first n odd integers is n^2. To show: $\forall n\, P(n)$, where

$$P(n) \text{ is } 1 + 3 + \cdots + (2n-1) = n^2$$

Example 2: Sum of first n perfect squares. To show $\forall n\, P(n)$, where

$$P(n) \text{ is } 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Example 3: If $n \geq 1$ is odd then $8 \mid n^2 - 1$. To show: $\forall m \geq 0\ P(m)$, where

$$P(m) \text{ is } 8 \mid (2m+1)^2 - 1$$

Fermat's Little Theorem

Theorem: If p is prime and $r \geq 0$ then $p \mid r^p - r \quad (*)$

Binomial Theorem: For all $a, b \in \mathbb{R}$ and $n \geq 0$

$$(a+b)^n \;=\; \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Harmonic Numbers

Definition: The nth harmonic number is the sum

$$H_m \;=\; 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$$

Fact: For each $n \geq 0$, $H_{2^n} \geq 1 + n/2$

Corollaries:

- H_n tends to infinity as n tends to infinity
- $H_n \geq 1 + \lfloor \log_2 n \rfloor / 2$ for each $n \geq 1$

Theorem: $H_n - \ln n \to \gamma = .577...$ (Euler's constant) as $n \to \infty$

An Inequality

Fact: If $x_1, \ldots, x_n$ are numbers between 0 and 1 then

$$1 - \sum_{i=1}^{n} x_i \;\leq\; \prod_{i=1}^{n} (1 - x_i)$$

Corollary: Under the same conditions

$$1 - \sum_{i=1}^{n} x_i \;\leq\; \prod_{i=1}^{n} \sqrt{1 - x_i} \;\leq\; \prod_{i=1}^{n} \sqrt{1 - x_i^2}$$

Induction with a Stronger Inductive Hypothesis.

Given: Propositional function $P(n)$ with domain $\{1, 2, \ldots\}$

Basis step: Show that $P(1)$ is true

Inductive step: Show $P(1) \wedge \cdots \wedge P(k) \rightarrow P(k+1)$ is true for each $k \geq 1$

- Assume that $P(1) \wedge \cdots \wedge P(k)$ is true – "strong inductive hypothesis"
- Establish that $P(k+1)$ is true

Conclusion: $P(n)$ is true for every $n \geq 1$

Can view strong induction as a (new) rule of inference. (Validity follows from well-ordering principle.)

$$[P(1) \wedge \forall k\,(P(1) \wedge \cdots \wedge P(k) \rightarrow P(k+1))] \rightarrow \forall n\, P(n)$$

Ex. Prime Factorization

Thm: Every integer $n \geq 2$ can be written as a product of primes.

Proof: Strong induction. Propositional function: for $n \geq 2$

$$P(n) \;=\; n \text{ can be written as a product of primes}$$

Basis: $P(2)$ is true as 2 is prime.

Induction: Suppose that $P(2), P(3), \ldots, P(k)$ are true.

- *Case 1:* Suppose $k+1$ is prime
- *Case 2:* Suppose $k+1$ is composite

Ex. Piles of Stones

Given: Pile of $n \geq 2$ stones

- split pile into two piles of size $r, s \geq 1$ with $r + s = n$
- compute product rs of pile sizes
- continue splitting piles into smaller ones until every pile has one stone

Claim: No matter how piles split, sum of products rs over splits is $n(n-1)/2$

Proof: Strong induction. Propositional function: for $n \geq 2$

$$P(n) = \text{ starting with } n \text{ stones, sum of products is } n(n-1)/2$$

Basis: Consider $P(2)$

Induction: Suppose that $P(2), P(3), \ldots, P(k)$ are true

Basics of Counting

Product Rule

Product Rule: Suppose that the elements of a collection S can be specified by a sequence of k steps such that

- There are n_j possibilities at step j
- The selections made at steps $1, \ldots, j$ do not affect the *number* of possibilities at step $j + 1$

Then S has $n_1 \, n_2 \cdots n_k$ elements

Example: Cartesian product of sets $A_1, \ldots, A_k$ is

$$A_1 \times \cdots \times A_k \ = \ \{(a_1, \ldots, a_k) : a_1 \in A_1, \ldots, a_k \in A_k\}$$

By product rule $|A_1 \times \cdots \times A_k| \ = \ |A_1| \cdots |A_k|$

Example: Counting Functions

Given: Finite sets $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$

1. What is the number of functions $f : A \to B$?
2. What is the number of one-to-one functions $f : A \to B$?
3. What is the number of onto functions $f : A \to B$?

Indicator Functions

Definition: The *indicator function* of a proposition q is given by

$$I(q) = \begin{cases} 1 & \text{if } q \text{ is true} \\ 0 & \text{if } q \text{ is false} \end{cases}$$

Example: Find the size of 2^S for a finite set $S = \{s_1, \ldots, s_n\}$

- Define function $f : 2^S \to \{0,1\}^n$ from 2^S to binary n-tuples by

$$f(A) \;=\; (I(s_1 \in A), I(s_2 \in A), \ldots, I(s_n \in A))$$

- Can check that $f(\cdot)$ is one-to-one and onto, so

$$|2^S| \;=\; |\{0,1\}^n| \;=\; 2^n \;=\; 2^{|S|}$$

Sum Rule

Sum Rule: S'pose that each element of a collection S is one of k types, and

- There are n_j elements of type j
- No element can be of more than one type.

Then $|S| = n_1 + \cdots + n_k$

Equivalent Form: If $S = A_1 \cup \cdots \cup A_k$ where $A_i \cap A_j$ for $i \neq j$ then

$$|S| = |A_1| + \cdots + |A_k|$$

Example: How many binary sequences b of length 6 begin with 01 or 001?

Inclusion-Exclusion

Fact: If A, B are sets then $|A \cup B| = |A| + |B| - |A \cap B|$.

General Form: For sets $A_1, \ldots, A_n$

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \cdots i_k \leq n} |A_{i_1} \cap \cdots \cap A_{i_k}|$$

Example: How many sequences $b \in \{0,1\}^8$ begin with 00 or end with 11?

Example: Suppose you have 5 friends who play golf, 8 who play tennis, and 3 who play both. How many of your friends play golf or tennis?

The Pigeonhole Principle

Fact: If $k+1$ objects are placed in k boxes then one box must contain at least two objects.

Why? Let $n_j = \#$ objects in box j. If each $n_j \leq 1$ then

$$\sum_{j=1}^{k} n_j \;\leq\; \sum_{j=1}^{k} 1 \;=\; k \;<\; k+1.$$

Examples

1. Among 13 people, at least two have their birthday in the same month.
2. Among 11 people, at least two have the same last digit in their phone number.
3. If $f : A \to B$ is a function and $|B| < |A|$ then f is not one-to-one

More Elaborate Applications of PHP

Fact: At a party with $n \geq 2$ guests there are at least two people with the same number of friends. (Assume a is friends with b iff b is friends with a.)

Why? For $1 \leq j \leq n$ let $m_j = \#$ friends of guest j at the party

- *Case 1:* Everybody knows somebody. Then each of $m_1, \ldots, m_n$ is between 1 and $n-1$, so two of these numbers must be the same

- *Case 2:* Somebody has no friends. Then each of $m_1, \ldots, m_n$ is between 0 and $n-2$, so two of these numbers must be the same

Triathlon Training (adapted from website of P. Talwalkar)

Know: Gary is training for a triathlon. Over a 30 day period he trains at least once every day, and 45 times in total.

Claim: There is a set of consecutive days when Gary trains exactly 14 times.

Why? For $j = 1, \ldots, 30$ let $s_j = \#$ workouts by end of day j. We know that

$$1 \leq s_1 < s_2 < \cdots < s_{29} < s_{30} = 45.$$

Adding 14 to each term gives $15 \leq s_1 + 14 < \cdots < s_{30} + 14 = 59$.

Upshot: The numbers $s_1, \ldots, s_{30}, s_1 + 14, \ldots, s_{30} + 14$ lie between 1 and 59

By PHP, two of these 60 numbers are the same, and as the numbers within each group are strictly increasing, there must be some i, j such that

$$s_i = s_j + 14 \quad \Leftrightarrow \quad s_i - s_j = 14$$

Another Application of PHP

Fact: For each $n \geq 1$ there is an $r \geq 1$ s.t. the decimal expansion of rn contains only 0s and 1s

Generalized Pigeon Hole Principle

Generalized PHP: If N objects are placed in k boxes, then there is a box containing at least $\lceil N/k \rceil$ objects

Example: Rolling a 6-sided die

1. How many rolls guarantee that we see some number at least 4 times?
2. How many rolls guarantee that we see 1 at least 2 times?

Example: Among a class of 60 people at least $\lceil 60/5 \rceil = 12$ will receive one of the letter grades A, B, C, D, F.