Introduction to Decision Sciences

Lecture 7

Andrew Nobel

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Summations

Summation Notation

Given: Numerical sequence $a_1, a_2, a_3, \ldots \in \mathbb{R}$

Definition: For $1 \leq m \leq n$ the sum of a_i as i goes from m to n is

$$\sum_{i=m}^{n} a_i \;=\; a_m + a_{m+1} + \cdots + a_{n-1} + a_n$$

Terminology

- i is the index of summation
- m is the lower limit of summation
- n is the upper limit of summation

Note: Choice of index i is not critical: $\sum_{i=m}^{n} a_i = \sum_{r=m}^{n} a_r$

Infinite sums

Given: Numerical sequence $a_1, a_2, a_3, \ldots \in \mathbb{R}$

Definition: Infinite sum of a_i as i goes from 1 to infinity is defined as the limit of finite sums (if the limit exists). Formally

$$\sum_{i=1}^{\infty} a_i \ := \ \lim_{n \to \infty} \sum_{i=1}^{n} a_i$$

Change of Index

Idea: Analogous to change of variables in integration.

Example: Let $f : \{1, \ldots, n\} \to \mathbb{R}$ be a sequence with sum $s = \sum_{k=1}^{n} f(k)$

Consider new index $j = k - 1$. Note that

- $k = j + 1$
- $k = 0 \Leftrightarrow j = 0$
- $k = n \Leftrightarrow j = n - 1$

Thus we can write sum s equivalently as $s = \sum_{j=0}^{n-1} f(j + 1)$

Some Sums

Fact: For $n \geq 1$ sum of first n positive numbers

$$\sum_{k=1}^{n} k \;=\; 1 + 2 + \cdots + n \;=\; \frac{(n+1)n}{2}$$

Harmonic series: The finite sum

$$\sum_{k=1}^{n} \frac{1}{k}$$

is approximately equal to $\log_e n$, and tends to infinity as n increases.

Geometric Series

Fact: If $x \neq 1$ then the nth term of the geometric series is

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}$$

Corollary: $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$

Corollary: If $|x| < 1$ then

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}$$

Extension: If $|x| < 1$ then

$$\sum_{k=1}^{\infty} k\, x^{k-1} = \frac{1}{(1 - x)^2}$$

Double Sums

Given: Array of numbers $M = \{a_{ij}\}$ with

- m rows $i = 1, \ldots, m$
- n columns $j = 1, \ldots, n$
- a_{ij} = entry in row i and column j

Definition: The double sum of a_{ij} is the sum of the mn entries of the array

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij}) = \sum_{i=1}^{m} \text{sum of entries in row } i$$

Properties of Double Sums

Fact: Order of summation is not important

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \;=\; \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}$$

Fact: Sum of products equals product of sums

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j \;=\; \left(\sum_{i=1}^{m} a_i \right) \left(\sum_{j=1}^{n} b_j \right)$$

Corollary: (Square of a sum) For every $a_1, \ldots, a_m \in \mathbb{R}$

$$0 \;\leq\; \left(\sum_{i=1}^{m} a_i \right)^2 \;=\; \sum_{i=1}^{m} \sum_{j=1}^{m} a_i \, a_j$$

Elegant argument

Fact: Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be real numbers. Then

$$\sum_{i=1}^{m} \sum_{j=1}^{m} \frac{a_i\, a_j}{i+j} \geq 0$$

Proof: Note that for every $0 \leq x \leq 1$ we have

$$0 \;\leq\; \left(\sum_{i=1}^{m} a_i\, x^{i-1/2} \right)^2 \;=\; \sum_{i=1}^{m} \sum_{j=1}^{m} a_i\, a_j\, x^{i+j-1}$$

Integrate both sides of the last inequality. Use the fact that integration is linear and that

$$\int_0^1 x^{\alpha}\, dx = 1/(\alpha+1)$$

Recursive Sequence

Recursive sequence of order $k \geq 1$ defined in two parts

- Specify k initial values $a_0, \ldots, a_{k-1}$
- Recursive relation: For $n \geq k$ express a_n in terms of k previous values $a_{n-1}, \ldots, a_{n-k}$, and possibly n

Note: Using the initial values and recursive relation, we can find every term a_n of the sequence.

Note: To find a closed form for a recursive series, work backwards using the definition.

Examples

Example 1: Recursive sequence $a_0 = 1$ and $a_n = a_{n-1} + 3$

Claim: $\{a_n\}$ is linear series with closed form $a_n = 1 + 3n$ for $n \geq 0$

Example 2: Recursive sequence with $a_0 = 5$ and $a_n = 2 \cdot a_{n-1}$

Claim: $\{a_n\}$ is geometric series with closed form $a_n = 5 \cdot 2^n$ for $n \geq 0$

Example 3: Recursive sequence with $a_0 = 1$ and $a_n = na_{n-1}$

Claim: $\{a_n\}$ is factorial series with closed form $a_n = n!$ for $n \geq 0$

Examples, cont.

[Finding closed form may require fancier mathematical tools]

Example 4: Recursive sequence with $a_0 = 1$, $a_1 = 4$ and $a_n = a_{n-1} - 2a_{n-2}$

Example 5 (Fibonacci Series): Sequence $\{a_n : n \geq 0\}$ specified by

- $a_0 = a_1 = 1$
- $a_n = a_{n-1} + a_{n-2}$

Closed form turns out to be

$$a_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n + \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

Note that $(1 \pm \sqrt{5})/2$ are the roots of quadratic equation $r^2 - r - 1 = 0$.

Cardinality

Cardinality

Recall: If A is a finite set, then cardinality $|A| =$ number of elements in A

Fact: If A, B are finite, then $|A| = |B|$ iff there exists a bijection $f : A \to B$

Definition

- Two sets A and B (possibly infinite) have the same cardinality if there exists a bijection $f : A \to B$
- A set A is countable if it is finite or has the same cardinality as $\mathbb{N}$
- A set A is uncountable if it is not countable

Examples

Example: The integers $\mathbb{Z}$ are countable

Example: The rationals $\mathbb{Q}$ are countable

Example: The interval $[0, 1) = \{x : 0 \leq x < 1\}$ is uncountable

Example: The real numbers $\mathbb{R}$ are uncountable

Divisibility

Factors and Multiples

Given: $a, b \in \mathbb{Z}$ with $a \neq 0$.

Definition: We say a divides b written $a|b$ if $b = ac$ for some $c \in \mathbb{Z}$.

- a is a *factor* of b
- b is a *multiple* of a

Fact: Let $a, b, c \in \mathbb{Z}$

- If $a|b$ and $a|c$ then $a|(b + c)$
- If $a|b$ and $b|c$ then $a|c$
- If $a|b$ then $a|bc$ for all $c \in \mathbb{Z}$

Cor: If $a|b$ and $a|c$ then $a|mb + nc$ for all $m, n \in \mathbb{Z}$

Division Algorithm

Fact: Let $d \geq 1$. If $a \in \mathbb{Z}$ then there exists a unique *quotient* $q \in \mathbb{Z}$ and *remainder* $0 \leq r < d$ such that

$$a \;=\; q\,d + r \tag{0.1}$$

Proof: For each $k \in \mathbb{Z}$ define

$$A_k = \{k\,d + r : 0 \leq r \leq d-1\} = \text{ integers between } kd \text{ and } (k+1)d - 1$$

Note that $\mathbb{Z} = \bigcup_{k \in \mathbb{Z}} A_k$ and $A_i \cap A_j = \emptyset$ if $i \neq j$.

Definition: When (0.1) holds, write $r = a \bmod d$. Terminology: r equals a modulo d, meaning that r is the remainder when a is divided by d.