Introduction to Decision Sciences

Lecture 6

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Functions

Functions

Given: Sets A and B, possibly different

Definition: A *function* $f : A \to B$ is a rule that assigns every element $a \in A$ to a unique element $f(a) \in B$.

- A called the *domain* of f
- B called the *range* of f

In general

- range A, domain B can be finite or infinite, and need not be numerical
- changing the range A or domain B changes the function

Some Common Real-Valued Functions

A. Defined for all real arguments

1. Constants $f(x) = 1$
2. Linear (affine) functions $f(x) = ax + b$
3. Polynomials $f(x) = \sum_{k=0}^{d} a_k x^k$
4. Exponential function $f(x) = e^{ax}$
5. Sine function $f(x) = \sin(x)$ (also cosine, tangent)
6. Other $f(x) = e^{-x^2/2}$

B. Defined for non-negative/positive arguments

1. Square root $f(x) = \sqrt{x}$ (also cube roots, fourth roots, and so on).
2. Logarithm $f(x) = \log x$ (usually base 10)
3. Other $f(x) = x \log x$

Image and Pre-Image

Definition: Let $f : A \to B$ be a function

- The *image* of $S \subseteq A$ under f is

$$f(S) = \{f(s) : s \in S\} \subseteq B \quad \text{(think pushforward)}$$

- The *pre-image* of $T \subseteq B$ under f is

$$f^{-1}(T) = \{a : f(a) \in T\} \subseteq A \quad \text{(think pullback)}$$

Note: pre-image $f^{-1}(T)$

- is well defined even if f is not invertible in the usual sense
- answers the question "when is $f(a)$ in T?"

One-to-One, Onto, Bijection

Definition: Let $f : A \to B$ be a function

- f is 1:1 (injective) if distinct points in A get mapped to distinct points in B

$$\forall a_1, a_2 \in A \ \ [a_1 \neq a_2 \to f(a_1) \neq f(a_2)]$$

- f is onto (surjective) if every point in B is the image of some point in A

$$\forall\, b \in B \ \ \exists\, a \in A \ \ f(a) = b$$

- f is a bijection if it is 1:1 and onto

Inverse of a Function

Fact: If $f : A \to B$ is a bijection then for every $b \in B$ there is a unique $a \in A$ such that $f(a) = b$.

Definition: If $f : A \to B$ is a bijection, define the *inverse* $f^{-1} : B \to A$ by

$$f^{-1}(b) = \text{ unique } a \in A \text{ s.t. } f(a) = b$$

Fact

- For each $a \in A$, $f^{-1}(f(a)) = a$
- For each $b \in B$, $f(f^{-1}(b)) = b$

Increasing and Decreasing Functions

Given: Function $f : A \to B$ with $A, B \subseteq \mathbb{R}$

Definition

- f is *increasing* if for all $x, y \in A$, $x \leq y$ implies $f(x) \leq f(y)$.
- f is *strictly increasing* if for all $x, y \in A$, $x < y$ implies $f(x) < f(y)$.
- f is *decreasing* if for all $x, y \in A$, $x \leq y$ implies $f(x) \geq f(y)$.
- f is *strictly decreasing* if for all $x, y \in A$, $x < y$ implies $f(x) > f(y)$.

Fact: If f is strictly increasing (or decreasing) then f is 1:1

Composition of Functions

Definition: The composition of two functions $g : A \to B$ and $f : B \to C$ is the function $f \circ g : A \to C$ defined by

$$(f \circ g)(a) = f(g(a))$$

Definition: The identity function $i_A : A \to A$ is defined by $i_A(a) = a$.

Definition: Given $f, g : A \to \mathbb{R}$ define

- sum $f + g : A \to \mathbb{R}$ by $(f + g)(a) = f(a) + g(a)$
- product $fg : A \to \mathbb{R}$ by $(fg)(a) = f(a)\, g(a)$

Floor and Ceiling Functions

Definition: For $x \in \mathbb{R}$

- $\lfloor x \rfloor$ = largest integer less than or equal to x
- $\lceil x \rceil$ = smallest integer greater than or equal to x

Fact

- $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
- $\lfloor -x \rfloor = -\lceil x \rceil$, $\lceil -x \rceil = -\lfloor x \rfloor$
- $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$

Sequences

Sequences

Informal: A *sequence* is an ordered list (finite or infinite) of objects.

Formally: Given

- Index set $A \subseteq \mathbb{N}$, e.g., $A = \{0, 1, \ldots, N\}, \{1, 2, 3, \ldots\}$
- Set of objects S, e.g, numbers, names, sets, etc.

A *sequence* of objects in S indexed by A is a function $f : A \to S$

Notation/Terminology:

- For $n \in A$, let $s_n = f(n)$
- Denote a generic sequence by $(s_n : n \in A)$
- A numerical sequence often referred to as a series

Some Numerical Sequences

- **Harmonic:** Here $s_n = 1/n$ for $n \geq 1$

$$1,\ 1/2,\ 1/3,\ 1/4, \ldots$$

- **Geometric:** Here $s_n = ar^n$ for $n \geq 0$ with $a, r \in \mathbb{R}$

$$a,\ ar,\ ar^2,\ ar^3, \ldots$$

- **Arithmetic:** Here $s_n = a + nd$ for $0 \leq n \leq N$ with $a, d \in \mathbb{R}$

$$a, a + d, a + 2d, \ldots a + Nd$$

  Here a is the initial term, d is the common difference, and $N + 1$ is the length of the progression.

Extrapolation

Given: First few terms of sequence $\{s_n\}$

Goal: Find a generating rule compatible with these terms

Examples

- $s_0, \ldots, s_4$ equal to $1, 4, 7, 10, 13$
- $s_0, \ldots, s_4$ equal to $3, 6, 12, 24, 48$
- $s_1, \ldots, s_5$ equal to $2, 8, 26, 80, 242$

Summations

Summation Notation

Given: Numerical sequence $a_1, a_2, a_3, \ldots \in \mathbb{R}$

Definition: For $1 \leq m \leq n$ the sum of a_i as i goes from m to n is

$$\sum_{i=m}^{n} a_i \;=\; a_m + a_{m+1} + \cdots + a_{n-1} + a_n$$

Terminology

- i is the index of summation
- m is the lower limit of summation
- n is the upper limit of summation

Note: Choice of index i is not critical: $\sum_{i=m}^{n} a_i = \sum_{r=m}^{n} a_r$

Infinite sums

Given: Numerical sequence $a_1, a_2, a_3, \ldots \in \mathbb{R}$

Definition: Infinite sum of a_i as i goes from 1 to infinity is defined as the limit of finite sums (if the limit exists). Formally

$$\sum_{i=1}^{\infty} a_i \ := \ \lim_{n \to \infty} \sum_{i=1}^{n} a_i$$

Change of Index

Idea: Analogous to change of variables in integration.

Example: Let $f : \{1, \ldots, n\} \to \mathbb{R}$ be a sequence with sum $s = \sum_{k=1}^{n} f(k)$

Consider new index $j = k - 1$. Note that

- $k = j + 1$
- $k = 0 \Leftrightarrow j = 0$
- $k = n \Leftrightarrow j = n - 1$

Thus we can write sum s equivalently as $s = \sum_{j=0}^{n-1} f(j + 1)$

Some Sums

Fact: For $n \geq 1$ sum of first n positive numbers

$$\sum_{k=1}^{n} k \;=\; 1 + 2 + \cdots + n \;=\; \frac{(n+1)n}{2}$$

Harmonic series: The finite sum

$$\sum_{k=1}^{n} \frac{1}{k}$$

is approximately equal to $\log_e n$, and tends to infinity as n increases.

Geometric Series

Fact: If $x \neq 1$ then the nth term of the geometric series is

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}$$

Corollary: $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$

Corollary: If $|x| < 1$ then

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}$$

Extension: If $|x| < 1$ then

$$\sum_{k=1}^{\infty} k\, x^{k-1} = \frac{1}{(1 - x)^2}$$

Double Sums

Given: Array of numbers $M = \{a_{ij}\}$ with

- m rows $i = 1, \ldots, m$
- n columns $j = 1, \ldots, n$
- $a_{ij} =$ entry in row i and column j

Definition: The double sum of a_{ij} is the sum of the mn entries of the array

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij}) = \sum_{i=1}^{m} \text{sum of entries in row } i$$

Properties of Double Sums

Fact: Order of summation is not important

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}$$

Fact: Sum of products equals product of sums

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j = \left(\sum_{i=1}^{m} a_i\right)\left(\sum_{j=1}^{n} b_j\right)$$

Corollary: Square of a sum

$$0 \leq \left(\sum_{i=1}^{m} a_i\right)^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j$$