# Introduction to Decision Sciences

## Lecture 5

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# Maxima, Minima, Absolute Value

# Maxima and Minima

**Definition:** The *maximum* of $a$ and $b$ is the larger of the two numbers

$$\max(a, b) = \begin{cases} a & \text{if } a \geq b \\ b & \text{otherwise} \end{cases}$$

**Definition:** The *minimum* of $a$ and $b$ is the smaller of the two numbers

$$\min(a, b) = \begin{cases} a & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$$

# Maxima and Minima, basic properties

**Fact:** For any numbers $a, b$

(1) $a, b \leq \max(a, b)$

(2) $\min(a, b) \leq a, b$

(3) If $a, b \geq 0$ then $\max(a, b) \leq a + b$.

(4) $a + b = \max(a, b) + \min(a, b)$.

# Maxima and Minima for Finite Sequences

**Definition:** The *maximum* of a numerical sequence $a_1, a_2, \ldots, a_n \in \mathbb{R}$ is the largest element of the sequence

$$\max_{1 \leq i \leq n} a_i = \; a_j \text{ such that } a_j \geq a_i \text{ for } 1 \leq i \leq n$$

**Note:** There is an analogous definition for the minimum of $a_1, a_2, \ldots, a_n$. Can you write this down?

**Fact:** If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are numerical sequences then

$$\max_{1 \leq i \leq n} (a_i + b_i) \;\leq\; \max_{1 \leq j \leq n} a_j \;+\; \max_{1 \leq k \leq n} a_k$$

# Absolute Value and Basic Properties

**Definition:** The *absolute value* of $x \in \mathbb{R}$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

**Fact:** For each $x \in \mathbb{R}$

(1) $|x| \geq 0$

(2) $|x| = \max(x, -x)$

(2) $|x|^2 = x^2$

(3) $|x| = \sqrt{x^2}$

# Absolute Values of Products and Sums

**Fact:** For $x, y \in \mathbb{R}$, we have $|xy| = |x||y|$

**Triangle inequality:** For $x, y \in \mathbb{R}$, we have $|x + y| \leq |x| + |y|$

**Interpretation:** The distance between numbers $x, y$ is usually measured by $|x - y|$. The triangle inequality implies that for every number $z$

$$|x - y| \leq |x - z| + |z - y|$$

Why is this inequality true? What does it say about distances?

# Fun with Squares

**Fact:** For every $a, b \in \mathbb{R}$, $2ab \leq a^2 + b^2$

**Corollary:** For every $a, b \in \mathbb{R}$, $(a + b)^2 \leq 2a^2 + 2b^2$

**Corollary:** For every $x, b \in \mathbb{R}$, $\sqrt{ab} \leq (a + b)/2$.

# Sets

# Sets

**Definition:** A *set* is an unordered collection of distinct objects. Members of a set are called *elements*.

- Sets denoted by $A, B, C, U, V$
- Elements denoted by $x, y, u, v$
- Membership: $x \in A$ means $x$ is an element of $A$

**Specifying sets**

- Finite or infinite list: $A = \{$Bob, Nancy, Elaine$\}$ or $U = \{1, 2, 3, \ldots\}$
- Description: $V = \{x : x$ is a prime number less than 100$\}$

Note: Read $\{x :$ ...$\}$ as “the set of all $x$ such that ... holds.

# Cardinality

**Definition**

- A set $A$ is *finite* if it has finitely many elements. Otherwise $A$ is *infinite*.
- The *cardinality* of a finite set $A$, denoted by $|A|$, is the number of elements in $A$.

**Definition:** The empty set $\emptyset$ is the set with no elements. Note that $\emptyset$ is finite, with $|\emptyset| = 0$. We can write $\emptyset = \{\,\}$.

**Important:** Elements of sets can be sets. For example

$$\{\} = \emptyset,\ \{\emptyset\},\ \{\{\emptyset\}\},\ \{\emptyset, \{\emptyset\}\}$$

# Equality, Containment, Power Sets

**Definition** Let $A$ and $B$ be sets

- $A = B$ if $A$ and $B$ have the same elements
- $A \subseteq B$ if every element of $A$ is an element of $B$

**Note:** $A = B$ equivalent to $A \subseteq B$ and $B \subseteq A$

**Logic**

- $A = B$ equivalent to $\forall x\ (x \in A \leftrightarrow x \in B)$
- $A \subseteq B$ equivalent to $\forall x\ (x \in A \rightarrow x \in B)$

**Definition** The *power set* of a set $A$, denoted $P(A)$ or $2^A$, is the set of all subsets of $A$. If $A$ finite then $|2^A| = 2^{|A|}$.

# Ordered Pairs and Cartesian Products

**Notation:** $(a, b)$ denotes the *ordered pair* with $a$ in the first position and $b$ in the second position. Note that ordered pair $(a, b)$ is different from set $\{a, b\}$.

**Definition:** The Cartesian product of two sets $A$ and $B$ is

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

No restrictions: every $a \in A$ is paired with every $b \in B$

**Fact:** If $A$ and $B$ are finite then $|A \times B| = |A| \cdot |B|$

**Higher Order Products:** Cartesian product of sets $A_1, \ldots, A_n$ is

$$A_1 \times \cdots \times A_n = \{(x_1, \ldots, x_n) : x_i \in A_i\}$$

# Set Operations

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**Given:** Universal set $U$, subsets $A, B \subseteq U$.

**Basic Operations**

- $\overline{A} = \{x : x \notin A\}$ complement of $A$
- $A \cup B = \{x : x \in A \vee x \in B\}$ union of $A$ and $B$
- $A \cap B = \{x : x \in A \wedge x \in B\}$ intersection of $A$ and $B$

**Definition:** Sets $A$ and $B$ are *disjoint* if $A \cap B = \emptyset$

**Other Operations**

- $A \setminus B = A \cap B^c = \{x : x \in A \wedge x \notin B\}$ set difference
- $A \triangle B = (A \setminus B) \cup (B \setminus A)$ symmetric difference

# Basic Facts

**Fact:** Given subsets $A, B$ of a universal set $U$

- $A \cap B \subseteq A, B \subseteq A \cup B$
- $A \cap \overline{A} = \emptyset$, $A \cup \overline{A} = U$
- $\overline{A} = U \setminus A$
- $\overline{\overline{A}} = A$

**Inclusion Exclusion:** Given finite sets $A, B \subseteq U$

- If $A \cap B = \emptyset$ then $|A \cup B| = |A| + |B|$
- In general, $|A \cup B| = |A| + |B| - |A \cap B|$, so $|A \cup B| \leq |A| + |B|$

# Identities for Set Operations

**Given:** $A, B, C \subseteq U$

**Distributive Laws**

(1) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(2) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

**De Morgan's Laws**

(3) $\overline{A \cup B} = \overline{A} \cap \overline{B}$

(4) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

**Note:** Set identities parallel logical equivalences with correspondences

complement $\Leftrightarrow \neg$ $\quad$ $\cap \Leftrightarrow \wedge$ $\quad$ $\cup \Leftrightarrow \vee$

To prove set identities: treat assertion $x \in A$ as a logical proposition, then apply rules of logic.

# Multiple Unions and Intersections

**Definition:** Given subsets $A_1, A_2, A_3, \ldots$ of a universal set $U$

- The union of $A_1, A_2, A_3, \ldots$ is

$$\bigcup_{i=1}^{\infty} A_i = \{x : \exists i \geq 1 \text{ such that } x \in A_i\}$$

- The intersection of $A_1, A_2, A_3, \ldots$ is

$$\bigcap_{i=1}^{\infty} A_i = \{x : \forall i \geq 1 \text{ such that } x \in A_i\}$$

# Functions

# Functions

**Given:** Sets $A$ and $B$, possibly different

**Definition:** A *function* $f : A \to B$ is a rule that assigns every element $a \in A$ to a unique element $f(a) \in B$.

- $A$ called the *domain* of $f$
- $B$ called the *range* of $f$

**In general**

- range $A$, domain $B$ can be finite or infinite, and need not be numerical
- changing the range $A$ or domain $B$ changes the function

# Some Common Real-Valued Functions

**A.** Defined for all real arguments

1. Constants $f(x) = 1$

2. Linear (affine) functions $f(x) = ax + b$

3. Polynomials $f(x) = \sum_{k=0}^{d} a_k x^k$

4. Exponential function $f(x) = e^{ax}$

5. Sine function $f(x) = \sin(x)$ (also cosine, tangent)

6. Other $f(x) = e^{-x^2/2}$

**B.** Defined for non-negative/positive arguments

1. Square root $f(x) = \sqrt{x}$ (also cube roots, fourth roots, and so on).

2. Logarithm $f(x) = \log x$ (usually base 10)

3. Other $f(x) = x \log x$,

# Image and Pre-Image

**Definition:** Let $f : A \to B$ be a function

- The *image* of $S \subseteq A$ under $f$ is

$$f(S) = \{f(s) : s \in S\} \subseteq B \quad \text{(think pushforward)}$$

- The *pre-image* of $T \subseteq B$ under $f$ is

$$f^{-1}(T) = \{a : f(a) \in T\} \subseteq A \quad \text{(think pullback)}$$

**Note:** pre-image $f^{-1}(T)$

- is well defined even if $f$ is not invertible in the usual sense
- answers the question "when is $f(a)$ in $T$?"

# One-to-One, Onto, Bijection

**Definition:** Let $f : A \to B$ be a function

- $f$ is 1:1 (injective) if distinct points in $A$ get mapped to distinct points in $B$

$$\forall a_1, a_2 \in A \;\; [a_1 \neq a_2 \to f(a_1) \neq f(a_2)]$$

- $f$ is onto (surjective) if every point in $B$ is the image of some point in $A$

$$\forall\, b \in B \;\; \exists\, a \in A \;\; f(a) = b$$

- $f$ is a bijection if it is 1:1 and onto

# Inverse of a Function

**Fact:** If $f : A \to B$ is a bijection then for every $b \in B$ there is a unique $a \in A$ such that $f(a) = b$.

**Definition:** If $f : A \to B$ is a bijection, define the inverse $f^{-1} : B \to A$ by

$$f^{-1}(b) = \text{ unique } a \in A \text{ s.t. } f(a) = b$$

**Fact**

- For each $a \in A$, $f^{-1}(f(a)) = a$
- For each $b \in B$, $f(f^{-1}(b)) = b$

# Increasing and Decreasing Functions

**Given:** Function $f : A \to B$ with $A, B \subseteq \mathbb{R}$

**Definition**

- $f$ is *increasing* if for all $x, y \in A$, $x \leq y$ implies $f(x) \leq f(y)$.
- $f$ is *strictly increasing* if for all $x, y \in A$, $x < y$ implies $f(x) < f(y)$.
- $f$ is *decreasing* if for all $x, y \in A$, $x \leq y$ implies $f(x) \geq f(y)$.
- $f$ is *strictly decreasing* if for all $x, y \in A$, $x < y$ implies $f(x) > f(y)$.

**Fact:** If $f$ is strictly increasing (or decreasing) then $f$ is 1:1

# Composition of Functions

**Definition:** The composition of two functions $g : A \to B$ and $f : B \to C$ is the function $f \circ g : A \to C$ defined by

$$(f \circ g)(a) = f(g(a))$$

**Definition:** The identity function $i_A : A \to A$ is defined by $i_A(a) = a$.

**Definition:** Given $f, g : A \to \mathbb{R}$ define

- sum $f + g : A \to \mathbb{R}$ by $(f + g)(a) = f(a) + g(a)$
- product $fg : A \to \mathbb{R}$ by $(fg)(a) = f(a)\, g(a)$

# Floor and Ceiling Functions

**Definition:** For $x \in \mathbb{R}$

- $\lfloor x \rfloor$ = largest integer less than or equal to $x$
- $\lceil x \rceil$ = smallest integer greater than or equal to $x$

**Fact**

- $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
- $\lfloor -x \rfloor = -\lceil x \rceil$, $\lceil -x \rceil = -\lfloor x \rfloor$
- $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$