Introduction to Decision Sciences

Lecture 4

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Introduction to Proofs

Theorems and Proofs

Definition: A *theorem* is a true mathematical statement. The argument establishing the truth of a theorem is called a *proof*. Standard forms:

A. Proposition p (Ex: $\sqrt{2}$ is irrational.)

- Direct: establish truth of p in a direct manner
- Contradiction: assume $\neg p$ and derive a contradiction

B. Implication $p \to q$ (Ex: If m, n are odd, so is mn.)

- Direct: assume p and then show q
- Contraposition: assume $\neg q$ and then show $\neg p$

C. Biconditional $p \leftrightarrow q$ (Ex: n^2 is even if and only if n is even.)

- Direct: establish chain of equivalences between p and q
- First show $p \to q$ then show $q \to p$

Terminology Used in Mathematical Practice

- A *Theorem* is major or important result
- A *Proposition* is minor, less important result
- A *Lemma* is a supporting result used in the proof of a theorem or proposition
- A *Corollary* is an immediate or easy consequence of a theorem or proposition

Odd, Even, and Rational Numbers

Notation:

- Positive Integers $\mathbb{N}_+ = \{1, 2, \ldots\}$
- Natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$
- Integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Rational numbers $\mathbb{Q} = \{a/b : a, b \in \mathbb{Z}$ and $b \neq 0\}$
- Real numbers $\mathbb{R} = (-\infty, \infty)$

Definition

- An integer n is *even* if $n = 2k$ for some $k \in \mathbb{Z}$
- An integer n is *odd* if $n = 2k + 1$ for some $k \in \mathbb{Z}$

Products of Odd and Even numbers

Fact: The product of two odd integers is odd.

- Approach: direct argument from definition

Corollary: If n is odd then n^2 is odd.

- Special case of previous fact

Fact: If n^2 is even then n is even.

- Approach: Contraposition

An Equivalence Theorem

Fact: If n is an integer then the following statements are equivalent

(1) n is even

(2) $n + 1$ is odd

(3) n^2 is even

Goal: We wish to establish the truth of all propositions

$$(i) \rightarrow (j) \text{ where } i, j \text{ can be } 1, 2, \text{ or } 3.$$

Approach

- show that $(1) \leftrightarrow (2)$
- show that $(1) \leftrightarrow (3)$
- establish $(2) \leftrightarrow (3)$ using tautology $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r) \equiv T$.

Proof by Contradiction

Goal: Establish proposition p

Idea: Assume p is false and derive a contradiction

Formally

- Establish truth of $\neg p \to F$
- Conclude $\neg p$ is false, so p is true.

Fact: The square root of 2 is irrational.

Proof Methods and Strategy

Exhaustive proofs, proofs by cases

A. Exhaustive proof: Sufficient to consider and check a small number of examples.

Fact: There is no solution in integers of the equation $x^2 + 2y^4 = 8$.

B. Proof by cases: To establish $p \to q$ express $p = p_1 \vee \cdots \vee p_k$ as a disjunction of cases p_j and then establish $p_j \to q$ for $j = 1, \ldots, k$.

Fact: The last digit of a perfect square is $0, 1, 4, 5, 6$, or 9

Existence Proof

Goal: Establish proposition of the form $\exists x\, P(x)$.

- Constructive: Exhibit x such that $P(x)$ is true.
- Non-constructive: Establish truth of $\exists x\, P(x)$ without exhibiting a specific x for which $P(x)$ is true.

Fact: There is an irrational number x such that x^x is rational.

Fact: If the average $\overline{a}$ of n numbers $a_1, \ldots, a_n$ is greater than α, then at least one of the numbers is greater than α.

Inequalities

Preliminaries

Recall: Real number $\mathbb{R} = (-\infty, \infty)$, also called the real line.

Standard terminology: A real number x is

- positive if $x > 0$
- non-negative if $x \geq 0$
- negative if $x < 0$

Signs of Sums

Basic Properties 1: The sum of

- two positive numbers is positive
- two non-negative numbers is non-negative
- two negative numbers is negative

Signs of Products

Basic Properties 2: The product of

- two positive numbers is positive
- two non-negative numbers is non-negative
- two negative numbers is positive
- a positive number and a negative numbers is negative
- any number with zero is zero

Fact: For every number a we have $a^2 \geq 0$, and if $a \neq 0$ then $a^2 > 0$.

Inequalities

Basic Definition: For numbers $a, b \in \mathbb{R}$

(1) $a \leq b$ if $b - a \geq 0$ (can also write $b \geq a$)

(2) $a < b$ if $b - a > 0$ (can also write $b > a$)

Transitivity: If $a \leq b$ and $b \leq c$ then $a \leq c$.

Inequalities for Sums

Fact:

- If $a \leq b$ and $c \leq d$ then $a + c \leq b + d$.
- If $a < b$ and $c \leq d$ then $a + c < b + d$.

Corollary:

- If $a \leq b$ then $a + c \leq b + c$ for every c
- If $a < b$ then $a + c < b + c$ for every c

Corollary:

- If $a \leq 0$ then $a + c \leq c$ for every c
- If $0 \leq b$ then $c \leq b + c$ for every c

Inequalities for Products

Fact: If $a \leq b$ and $c \leq d$ then $ac \leq bd$.

Fact: Suppose that $a \leq b$.

- If $\alpha \geq 0$ then $\alpha a \leq \alpha b$
- If $\alpha \leq 0$ then $\alpha b \leq \alpha a$

Example: If $a \leq b$ then $-b \leq -a$.

Maxima, Minima, Absolute Values

Maxima and Minima

Definition: The *maximum* of a and b is the larger of the two numbers

$$\max(a,b) = \begin{cases} a & \text{if } a \geq b \\ b & \text{otherwise} \end{cases}$$

Definition: The *minimum* of a and b is the smaller of the two numbers

$$\min(a,b) = \begin{cases} a & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$$

Note: Both definitions extend to finite lists of numbers $a_1, a_2, \ldots, a_n$.

Maxima and Minima, basic properties

Fact: For any numbers a, b

(1) $a, b \leq \max(a, b)$

(2) $\min(a, b) \leq a, b$

(3) If $a, b \geq 0$ then $\max(a, b) \leq a + b$.

(4) $a + b = \max(a, b) + \min(a, b)$.

Absolute Value and Basic Properties

Definition: The *absolute value* of $x \in \mathbb{R}$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Fact: For each $x \in \mathbb{R}$

(1) $|x| \geq 0$

(2) $|x| = \max(x, -x)$

(3) $|x| = \sqrt{x^2}$

Corollary: For each $x \in \mathbb{R}$

(1) $x, -x \leq |x|$

(2) $|x|^2 = x^2$

Absolute Values of Products and Sums

Fact: For $x, y \in \mathbb{R}$, we have $|xy| = |x||y|$

Triangle inequality: For $x, y \in \mathbb{R}$, we have $|x + y| \leq |x| + |y|$

Interpretation: The distance between numbers x, y is usually measured by $|x - y|$. The triangle inequality implies that for every number z

$$|x - y| \leq |x - z| + |z - y|$$

Why is this inequality true? What does it say about distances?