Introduction to Decision Sciences

Lecture 3

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Predicates and Quantifiers

Predicates

Definition: A *domain* U is a set of objects x of interest.

Definition: A *predicate* P for U is a property of the objects $x \in U$ such that every $x \in U$ has P, or does not have P, but not both.

Think: Predicate $P \Leftrightarrow$ set of $x \in U$ having property P

Propositional function: If P is a predicate and x is in U then

$$P(x) = \text{ proposition "object } x \text{ has property } P\text{"}$$

Thus $P(x)$ is T if x has property P, and $P(x)$ is F otherwise.

Examples of Predicates

Example: Domain $U =$ all undergraduates at UNC

- Predicate P is the property of being a Senior
- Predicate Q is property of being enrolled in STOR 215
- $P(\text{Bob}) =$ "Bob is a Senior at UNC"
- $Q(\text{Kelly}) =$ "Kelly is enrolled in STOR 215"

Examples of Predicates

Example: Even numbers

- Set $U =$ positive integers $\{1, 2, 3, \ldots\}$
- Predicate P is the property of being even
- $P(x) =$ "x is even"

Example: Perfect squares

- Set $U =$ positive integers $\{1, 2, 3, \ldots\}$
- Predicate Q is the property of being a perfect square
- $Q(x) =$ "x is a perfect square"

More Exotic Example

Example: Pythagorean triples

- Set $U =$ all triples (x, y, z) of positive integers
- Predicate P is the property of being a Pythagorean triple
- $P(x)$ is the statement $x^2 + y^2 = z^2$

Quantifiers

Two flavors

- Universal: $\forall$ means "for all"
- Existential: $\exists$ means "there exists"

Definition: Let P be a predicate on a domain U

- $\forall x P(x)$ is a proposition that is T if and only if $P(x)$ is T *for each* x in U
- $\exists x P(x)$ is a proposition that is T if and only if $P(x)$ is T *for some* x in U

In other words

- $\forall x P(x)$ is true if every element of U has property P
- $\exists x P(x)$ is true if some element of U has property P

Quantifiers, cont.

Note: If domain $U = \{x_1, \ldots, x_n\}$ is finite, then

- $\forall x\, P(x) \equiv P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$
- $\exists x\, P(x) \equiv P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$

Note: Truth of $\forall x\, P(x)$ and $\exists x\, P(x)$ depends on domain U

- Domains: $U =$ positive integers, $V = \{2, 4, 6, 8\}$
- Predicates: $P =$ even, $Q =$ perfect square

Expressions with Quantifiers

Note: Quantifiers $\exists$, $\forall$ have higher precedence than other logical operations

Definitions: In an expression with quantifiers

- A variable x is *bound* if it is the subject of a quantifier, and *free* otherwise
- If all variables in an expression are bound, it is a proposition. Otherwise, it is a new predicate.
- The *scope* of a quantifier is the set of predicates to which it applies.

Example

- $\forall x \, (P(x) \vee Q(x))$
- $\forall x \, P(x) \vee Q(y)$

Logical Equivalence

Definition: Two propositions with quantifiers are logically equivalent if they have the same truth value, whatever the choice of predicates and domains.

Example

- $\forall x\, \forall y\, R(x, y) \equiv \forall y\, \forall x\, R(x, y)$ (and similarly for $\exists$)
- $\forall x\, (P(x) \land Q(x)) \equiv \forall x\, P(x) \land \forall x\, Q(x)$
- $\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$ (De Morgan)
- $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$ (De Morgan)

But note...

- $\forall x\, (P(x) \lor Q(x)) \not\equiv \forall x\, P(x) \lor \forall x\, Q(x)$
- $\exists x\, \forall y\, R(x, y) \not\equiv \forall y\, \exists x\, R(x, y)$

Translation with Operations and Quantifiers

General rules: Domain U with elements x

- $\forall =$ for all, all, every $x \in U$
- $\exists =$ there exists, there is, some, at least one $x \in U$
- $\wedge =$ and, $\vee =$ or, $\neg =$ not
- $p \to q =$ if p then q, or p is sufficient for q, or q is necessary for p

Translation Example

Domain $U =$ all students at UNC. Consider the following predicates

$$C = \text{taking STOR 215}$$

$$E = \text{speaks English}, \quad F = \text{speaks French}, \quad G = \text{speaks Greek}$$

- All students at UNC speak English
- Some UNC students are taking 215
- Every 215 student speaks French
- Some 215 student speaks Greek
- No 215 student speaks Greek
- Some 215 student speaks French and Greek
- Every 215 student who speaks French also speaks Greek
- Some 215 student does not speak French

Nested Quantifiers

Expressions with Nested Quantifiers

Given: Predicates $P(x, y)$ and $R(x, y, z)$

Examples

- $\forall x \, \exists y \, P(x, y)$
- $\exists x \, \forall y \, (P(x, y) \lor \exists z \, R(x, y, z))$

Note: these expressions are propositions (they have no free variables), so they are either T or F

Two Examples

A. Let $U = (-\infty, \infty)$ and $Q(x, y) : y^2 \geq 2x + 1$

- $\forall x\, \exists y\, Q(x, y)$
- $\exists x\, \forall y\, Q(x, y)$
- $\exists y\, \forall x\, Q(x, y)$
- $\forall x\, \forall y\, Q(x, y)$

B. Let $U =$ all students at UNC and $K(x, y) = x$ knows y

- $\exists y\, \forall x\, K(x, y)$
- $\forall x\, \exists y\, K(x, y)$
- $\forall x\, \forall y\, (K(x, y) \to K(y, x))$

Negation of Expressions with Nested Quantifiers

Arises in many mathematical arguments, e.g., proofs by contradiction and proofs by contraposition.

Idea: Successively apply De Morgan's laws for (i) quantifiers and (ii) compound propositions.

Example: Negation of $\forall x \, \exists y \, Q(x, y)$

Example: Negation of $\exists x \, \forall y \, (Q(x, y) \vee \exists z \, R(x, y, z))$

Translation with Nested Quantifiers

Example

- The product of two positive numbers is positive.
- Negation of this proposition.

Example

- Every positive number is the square of some non-zero number.
- Negation of this proposition.

Example

- Every positive number is the square of a unique positive number.
- Negation of this proposition.

Introduction to Proofs

Theorems and Proofs

Definition: A *theorem* is a true mathematical statement. The argument establishing the truth of a theorem is called a *proof*. Standard forms:

A. Proposition p (Ex: $\sqrt{2}$ is irrational.)

- Direct: directly establish truth of p
- Contradiction: assume $\neg p$ and derive a contradiction.

B. Implication $p \to q$ (Ex: If m, n are even, so is mn.)

- Direct: assume p and then show q
- Contraposition: assume $\neg q$ and then show $\neg p$

C. Biconditional $p \leftrightarrow q$ (Ex: n^2 is even if and only if n is even.)

- Direct: establish chain of equivalences between p and q
- First show $p \to q$ then show $q \to p$

Terminology Used in Mathematical Practice

- A *Theorem* is major or important result
- A *Proposition* is minor, less important result
- A *Lemma* is a supporting result, used in the proof of a theorem or proposition
- A *Corollary* is an immediate or easy consequence of a theorem or proposition

Odd, Even, and Rational Numbers

Notation:

- Positive Integers $\mathbb{N}_+ = \{1,\ 2, \ldots\}$
- Natural numbers $\mathbb{N} = \{0,\ 1,\ 2, \ldots\}$
- Integers $\mathbb{Z} = \{\ldots, -2,\ -1,\ 0,\ 1,\ 2, \ldots\}$
- Rational numbers $\mathbb{Q} = \{a/b : a, b \in \mathbb{Z}$ and $b \neq 0\}$
- Real numbers $\mathbb{R} = (-\infty, \infty)$

Definition

- An integer n is *even* if $n = 2k$ for some $k \in \mathbb{Z}$
- An integer n is *odd* if $n = 2k + 1$ for some $k \in \mathbb{Z}$

Products of Odd and Even numbers

Fact: The product of two odd integers is odd.

Corollary: If n is odd then n^2 is odd.

Fact: If n^2 is even then n is even.

Fact: Let n be any integer. The following statements are equivalent:

(1) n is even

(2) $n + 1$ is odd

(3) n^2 is even

Proof by Contradiction

Goal: Establish proposition p

Idea: Assume p is false and derive a contradiction

Formally

- Establish truth of $\neg p \to F$
- Conclude $\neg p$ is false, so p is true.

Fact: The square root of 2 is irrational.

Proof Methods and Strategy

Exhaustive proofs, proofs by cases

A. Exhaustive proof: Sufficient to consider and check a small number of examples.

Fact: There is no solution in integers of the equation $x^2 + 2y^4 = 8$.

B. Proof by cases: To establish $p \to q$ express $p = p_1 \vee \cdots \vee p_k$ as a disjunction of cases p_j and then establish $p_j \to q$ for $j = 1, \ldots, k$.

Fact: The last digit of a perfect square is $0, 1, 4, 5, 6$, or 9

Existence Proof

Goal: Establish proposition of the form $\exists x\, P(x)$.

- Constructive: Exhibit x such that $P(x)$ is true.
- Non-constructive: Establish truth of $\exists x\, P(x)$ without exhibiting a specific x for which $P(x)$ is true.

Fact: There is an irrational number x such that x^x is rational.

Fact: If the average $\overline{a}$ of n numbers $a_1, \ldots, a_n$ is greater than α, then at least one of the numbers is greater than α.