Introduction to Decision Sciences

Lecture 2

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Compound Proposition

A compound proposition is a combination of propositions using the basic operations. For example

- $\neg(p \wedge q)$
- $(\neg p) \vee (\neg q)$
- $\neg(p \vee q) \wedge q$

Conditional (If-Then) Statement

Implication: p implies q

- Denoted $p \to q$
- In words: "if p then q" (also "q if p", "p sufficient for q",...)
- Truth value: $p \to q$ is F only if p is T and q is F

Truth Table

Note: If p is F then $p \to q$ is T

Fact: $p \to q$ is logically equivalent to (has same truth values as) $\neg p \vee q$

Other Conditional Statements

Given: Propositions p and q, implication $p \to q$

- *Converse* of $p \to q$ is $q \to p$
- *Contrapositive* of $p \to q$ is $\neg q \to \neg p$
- *Biconditional* $p \leftrightarrow q$ defined by $(p \to q) \wedge (q \to p)$

Truth tables

Fact

- Contrapositive $\neg q \to \neg p$ equivalent to $p \to q$
- Biconditional $p \leftrightarrow q$ is T only when p, q have same truth values

Precedence of Operators

Notational device to save parentheses and simplify written expressions.

Add parentheses around elements of compound proposition in the following order of priority (first to last):

$$\neg \quad \wedge \quad \vee \quad \rightarrow \quad \leftrightarrow$$

Example: $\neg p \rightarrow q \wedge p$ is the same as $(\neg p) \rightarrow (q \wedge p)$.

Example: $\neg p \wedge q$ is the same as $(\neg p) \wedge q$.

When in doubt: use parentheses

Translation

Translation is the action of representing English sentences as logical expressions, and vice-versa

- Common activity when reading or proving theorems
- Enables one to check the validity of complex expressions where common sense may fail

Tautologies and Contradictions

Definition: A compound proposition is a

- *Tautology* if it is always true, whatever the truth values of its constituent propositions.
- *Contradiction* if it is always false, whatever the truth values of its constituent propositions.

Note: A single true proposition, e.g. $4 \geq 2$, is not considered to be a tautology. Likewise, a false proposition is not a contradiction.

Examples

- $p = q \vee \neg q$
- $p = q \wedge \neg q$

Logical Equivalence

Definition: Two compound propositions r and s are *logically equivalent*, written $r \equiv s$, if they have the same truth values, whatever the truth values of their constituent propositions.

Equivalently, $r \equiv s$ if $r \leftrightarrow s$ is a tautology

Examples

- $\neg(p \land q) \equiv \neg p \lor \neg q$
- $p \to q \equiv \neg p \lor q$

Equivalences can be verified by looking at truth tables, but for complex examples, it is helpful to have a set of simple rules that can be applied sequentially to an expression of interest.

Some notation: $T =$ tautology, $F =$ contradiction

Logical Equivalence

A. Unconditional Rules

- Identity: $p \wedge T \equiv p,\ p \vee F \equiv p$
- Domination: $p \wedge F \equiv F,\ p \vee T \equiv T$
- Idempotence: $p \wedge p \equiv p,\ p \vee p \equiv p$
- Double negation: $\neg(\neg p) \equiv p$
- Commutative: $p \vee q \equiv q \vee p,\ p \wedge q \equiv q \wedge p,$
- Associative: $p \vee (q \vee r) \equiv (p \vee q) \vee r$ and $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
- De Morgan: $\neg(p \wedge q) \equiv \neg p \vee \neg q$ and $\neg(p \vee q) \equiv \neg p \wedge \neg q$
- Absorbtion: $p \vee (p \wedge q) \equiv p$ and $p \wedge (p \vee q) \equiv p$

Logical Equivalences, continued

- Distinctive

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

B. Conditional Rules

- $p \to q \equiv \neg p \vee q$
- $p \to q \equiv \neg q \to \neg p$
- $p \vee q \equiv \neg p \to q$

C. Biconditional Rules

- $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
- $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

Predicates

Definition: A *domain* is a collection/set U of objects x of interest

Definition: A *predicate* P for U is a property of the objects $x \in U$ such that every $x \in U$ has P, or does not have P, but not both.

Note: Predicate equivalent to the subset of $x \in U$ having property P

Propositional function: If P is a predicate and x is in U let

$$P(x) = \text{ proposition “object } x \text{ has property } P\text{”}$$

Thus $P(x)$ is T if x has property P and F otherwise.

Examples of Predicates

Example: UNC Students

- Set $U =$ all undergraduates currently enrolled at UNC
- Predicate P is the property of being a Senior
- Predicate Q is property of being enrolled in STOR 215
- $P(\text{Bob}) =$ "Bob is a Senior at UNC"
- $Q(\text{Kelly}) =$ "Kelly is enrolled in STOR 215"

Examples of Predicates

Example: Even integers

- Set $U =$ positive integers $\{1, 2, 3, \ldots\}$
- Predicate P is the property of being even
- $P(x) =$ "x is even"

Example: Perfect squares

- Set $U =$ positive integers $\{1, 2, 3, \ldots\}$
- Predicate Q is the property of being a perfect square
- $P(x) =$ "x is a perfect square"

Quantifiers

Two flavors

- Universal: $\forall$ means "for all"
- Existential: $\exists$ means "there exists"

Definition: Let P be a predicate on a domain U

- $\forall x P(x)$ is a proposition that is T if and only if $P(x)$ is T *for each* x in U
- $\exists x P(x)$ is a proposition that is T if and only if $P(x)$ is T *for some* x in U

In other words

- $\forall x P(x)$ is true if every element of U has property P
- $\exists x P(x)$ is true if some element of U has property P

Quantifiers, cont.

Note: If domain $U = \{x_1, \ldots, x_n\}$ is finite, then

- $\forall x P(x) \equiv P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$
- $\exists x P(x) \equiv P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$

Note: Truth of $\forall x P(x)$ and $\exists x P(x)$ depends on domain U

Example

- Domains: $U =$ positive integers, $V = \{2, 4, 6, 8\}$
- Predicates: $P =$ even, $Q =$ perfect square